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A SINGULAR FREE BOUNDARY PROBLEM

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1. Introduction

In 1961 Chernoff [1] studied the problem of sequentially testing whether the drift of a Wiener process is positive or negative, given an a priori normal distribution, and showed that this problem can be reduced to a singular parabolic free boundary problem. A description of Chernoff's formulation and reduction of the problem can also be found in [7]. Briefly, one considers a Wiener–Levy stochastic process $\chi(\tau)$ and an associated process $\xi(\tau)$ with drift μ; i.e. $\xi(\tau)=\chi(\tau)+\mu\tau$ where μ is an unknown constant whose sign is to be determined.

μ is considered as a random variable with known a priori normal distribution. The problem then of observation and periodic testing to determine the sign of ξ and hypothesize the sign of μ in such a way as to minimize the expected cost of the operation becomes one of uniformly minimizing the Bayes risk $B(\xi, \tau)$. It is assumed that the cost of an incorrect decision is proportional to $|\mu|$ and that the cost of observation is constant per unit time. Chernoff then shows that B then satisfies the equation

$$\tfrac{1}{2}B_{\xi\xi}+\frac{\xi}{\tau}B_{\xi}+B_{\tau}+1=0$$

in the continuation region and certain boundary conditions as well. Then, defining a new function $u(x, t)$ in terms of the Bayes risk $B(\xi, \tau)$ and performing a change of variables Chernoff reduces the problem to the following singular parabolic free boundary problem: find a function $u(x, t)$ and a free boundary curve $x=s(t)$ such that

$$(\mathrm{P})\qquad \begin{aligned} &u_t-u_{xx}=-1/(2t^2) \quad \text{for} \quad 0<x<s(t), \qquad 0<t<T,\\ &u_x(0,t)=-\tfrac{1}{2} \quad \text{for} \quad 0<t<T,\\ &u(s(t),t)=u_x(s(t),t)=0 \quad \text{for} \quad 0<t<T,\\ &s(0)=0. \end{aligned}$$

It should be noted that the conditions on u_x are incompatible at the origin and that the equation is singular at $t=0$.

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