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SUMS OF GAUSS, EISENSTEIN, JACOBI, JACOBSTHAL, AND BREWER

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1. Introduction

In [1], we evaluated certain Gauss, Jacobi, and Jacobsthal sums over the finite field $GF(p)$, where p is an odd prime. One of the main objects of this paper is to evaluate such sums over $GF(p^2)$.

In Chapter 2, we give the basic theorems which relate the sums of Eisenstein, Gauss, Jacobi, and Jacobsthal. In Chapter 3, Jacobi sums associated with characters on $GF(p)$ of orders 5, 10, and 16 are evaluated, and the values of certain Jacobsthal sums over $GF(p)$ are determined. The formulae for these Jacobi sums and the Jacobi sums evaluated in [1] are utilized in Chapter 4, wherein we evaluate Jacobi and Eisenstein sums associated with characters on $GF(p^2)$ of orders 3, 4, 5, 6, 8, 10, 12, 16, 20, and 24. All of the evaluations in Chapters 3 and 4 are effected in terms of parameters that appear in the representations of the primes p as binary or quartic integral quadratic forms.

Many of the results of Chapters 3 and 4 are new, but some have been obtained elsewhere by the use of the theory of cyclotomic numbers. (In particular, see [7].) In contrast, our approach is via Jacobi and Eisenstein sums, as in [1] and [15]. For our purposes, this approach is perhaps simpler and more natural.

Another goal of this paper is to give a self-contained, systematic treatment of Brewer character sums. Several Brewer sums have been evaluated in the literature by a variety of methods. In Section 5.2, we develop a unified theory of Brewer sums. In particular, we express generalized Brewer sums $\Lambda_n(a)$ in terms of Jacobsthal sums over $GF(p)$ and Eisenstein sums, and so generalize a theorem of Robinson [23]. Our proofs do not depend upon the theory of cyclotomy, as do most existing proofs and explicit determinations. In Section 5.3, we apply our theory to give mostly new proofs of known formulae for $\Lambda_n(a)$ when $n = 1, 2, 3, 4, 5, 6, 8, 10$, and 12.

In Chapter 6, using primarily Theorem 2.7 and the formulae for Jacobi sums in Chapter 4, we evaluate certain Jacobsthal sums over $GF(p^2)$. In Chapter 7, using primarily Theorem 2.12 and the formulae for Eisenstein sums in Chapter 4, we evaluate the Gauss sums $\mathcal{G}_k = \sum_{\alpha \in GF(p^2)} e^{2\pi i \operatorname{tr}(\alpha^k)/p}$, for $k = 2, 3, 4, 6, 8$, and 12. Most of the results of Chapters 6 and 7 appear to be new.

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