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LIE ALGEBRA COHOMOLOGY AT IRREDUCIBLE MODULES

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We will develop a procedure for producing irreducible modules for the Lie algebra of a semisimple, simply connected algebraic group at which the 1-cohomology is non-zero. Further, we will relate our computations of Lie cohomology to the cohomology of the algebraic group. The cohomology of the group may be zero at a module where the cohomology of the Lie algebra is non-zero, but there is an efficient method for augmenting the module to give a module where the cohomology of the group is non-zero.

Hochschild showed that the (restricted) 1-cohomology of a non-abelian p-Lie algebra L is non-zero at the L-module Hom (LU_L, k), where U_L is the restricted universal enveloping algebra of L [3]. In Sections 1 and 2, we show that his methods can be used in the case of the Lie algebra of a Chevalley group to produce a good supply of irreducible modules $\{V_j\}$ at which the 1-Lie cohomology is non-zero. One begins with a suitable p-semi-linear map from the Lie algebra to the trivial Lie algebra k, and uses the isomorphism

$$H^2(LU_L, k) \cong H^1(LU_L, \operatorname{Hom}(LU_L, k))$$

to obtain a 1-cohomology class with values in Hom (LU_L, k). By passing to subquotients of Hom (LU_L, k), one obtains some irreducible modules $\{V_j\}$ at which the 1-cohomology is non-zero. The highest weights of these modules are the differentials of the elements $\{-\alpha_i\}$ where $\{\alpha_i\}$ is a basis for the root system of the group relative to a maximal torus T.

The space of 1-Lie cocycles at an irreducible module is itself a module for the group. In showing in Theorem 2.2 that the cohomology spaces are non-zero at $\{V_j\}_{j=1}^{\dim(T)}$, we produce a line of 1-cocycles that is stable under the action of an appropriate Borel subgroup of the group, and show that the weight of the line under the action of T does not occur in the module V_j. Consequently, the cocycles in the line are not coboundaries. At the same time, the weight of this line gives the highest weight of a composition factor of the 1-Lie cohomology as a module for the group. As an illustration, we give the result of Sections 1 and 2 specifically for the Lie algebra of the special linear group.

A module at which the group cohomology is non-zero may be obtained economically from V_j by tensoring V_j with the dual module $(H^1(L, V_j))^*$ (Corollary to Theorem 2.2).

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