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THE STRUCTURE OF MINIMA FOR BINARY QUADRATIC FORMS

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Abstract

In this paper, we construct examples of binary quadratic forms with positive, unattained minimum. To this end, we investigate the structure of the set of values taken by a certain class of indefinite forms.

1. Introduction

The work in this paper grew out of an attempt to answer a question posed by Paul Bateman: construct a binary quadratic form with a non-zero, unattained minimum. Throughout this paper, f will represent the binary form with real coefficients given by

$$f(x, y) = (x - \alpha y)(x - \beta y) \tag{1}$$

where α and β are both real or are complex conjugate. The minimum of f is defined as $\mu(f) = \inf |f(x, y)|$ taken over non-zero integer points (x, y).

Examples of the forms sought have been given by Schur (reported by Remak in [5]), and, independently of the author, by Larry Pinzur. Their examples were constructed by choosing α and β to have particular bounded continued fraction expansions. In Remak's terminology, the forms sought are ones that are not unimodularly equivalent to minimal forms. The form f, defined in (1), is a minimal form if $|f(x, y)| \geq 1$ for all non-zero integer points (x, y). Now, α and β have bounded continued fraction expansions if they are quadratic, that is, each is an irrational solution of a quadratic equation with rational coefficients. The continued fraction expansions of such numbers are periodic, therefore, bounded, Again, see [4] for details. Pinzur's examples, as the ones presented here, involve only quadratic numbers. In Schur's example, α is chosen to be quadratic while β has a bounded but non-periodic expansion. Throughout the rest of this paper, we assume that both α and β are quadratic.

$\mu(f)$ is an attained minimum if $\mu(f) = |f(x_0, y_0)|$ for some non-zero integer point (x_0, y_0). A number of ways of choosing α and β can be immediately eliminated as not answering Bateman's question. If α and β are not real, then one easily checks that $\mu(f)$ is indeed attained. If α is real, but is "too well" approximable by rationals, then $\mu(f) = 0$. By "too well" approximable, we mean that for any $\varepsilon > 0$, $|y|\,|x - \alpha y| < \varepsilon$ has infinitely many integer

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