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UNIVALENCE CRITERIA DEPENDING ON THE SCHWARZIAN DERIVATIVE[1]

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It is known [6, 7] that an analytic function f in $\Delta=\{z:|z|<1\}$ will be univalent if it satisfies certain conditions of the type

$$|\{f, z\}| \leq R(|z|), \tag{1}$$

where $\{f, z\}$ denotes the Schwarzian derivative

$$\{f, z\} \equiv \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2. \tag{2}$$

The most important such condition [6] is

$$|\{f, z\}| \leq \frac{2}{(1-|z|^2)^2} \tag{3}$$

This sufficient univalence criterion has two notable features: (a) If the constant 2 is replaced by 6, (3) is a necessary criterion for univalence [5], [6]. (b) If 2 is replaced by a smaller positive number, condition (3) guarantees, in addition, that f has a quasi-conformal extension to the entire complex plane [2], [3], [1].

Two other univalence criteria of the type (1) which, like (3), have the merit of using functions R of simple character are

$$|\{f, z\}| \leq \frac{\pi^2}{2} \quad [6] \tag{4}$$

and

$$|\{f, z\}| \leq \frac{4}{1-|z|^2} \quad [8] \tag{5}$$

In both cases the constants are the best possible. (For the sharpness of (3), cf. [4].)

The general procedure, described in [7], for generating univalence criteria of the type (1) relied heavily on the theory of linear second-order differential equations. In particular, the determination of the function R required the explicit knowledge of a solution of an associated differential equation. In the present paper we develop an alternative procedure which circumvents these difficulties. The only use of ideas properly belonging to the theory of differential equations will appear in the proof of the following lemma.

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