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$L(X)$ AS A SUBALGEBRA OF $K(X)^{**}$

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1. Introduction

For X and Y Banach spaces let $L(Y, X)$ and $K(Y, X)$ denote respectively the spaces of bounded and compact operators from Y into X. The relationship of $L(Y, X)$ and $K(Y, X)$ as Banach spaces has long been of interest. In some special cases, $L(Y, X)$ is actually equal to $K(Y, X)$ while in others $L(Y, X)$ is equal to $K(Y, X)^{**}$. See [8], [10] and [2], [5], [6], [11]. Recently, Jerry Johnson [9] has extended a weaker result in [6] and shown that if X has the bounded approximation property (metric approximation property), then $L(Y, X)$ can be imbedded isomorphically (isometrically) in $K(Y, X)^{**}$. The purpose of this paper is to study Johnson's imbedding for the case $Y = X$ in which $K(X)$ and $L(X)$ are Banach algebras.

For a Banach of Algebra $\mathcal{A}$, the Arens products (see Section 2) give two ways of regarding $\mathcal{A}^{**}$ as a Banach algebra so that the canonical image of $\mathcal{A}$ in $\mathcal{A}^{**}$ is subalgebra of $\mathcal{A}^{**}$. Specializing Johnson's imbedding to the case $Y = X$, it is natural to consider the operator induced multiplication on the image of $L(X)$ in $K(X)^{**}$. In Section 3, we discuss the imbedding of $L(X)$ into $K(X)^{**}$ under the assumption that X has the bounded approximation property and present an example in which *neither* Arens product coincides with operator induced multiplication. Hence, the imbedding need not be a Banach algebra isomorphism. In Section 4, under the assumption that $K(X)$ has a bounded two-sided weak identity, we show that the Johnson imbedding can be defined as a Banach algebra isomorphism, using the first Arens product on $K(X)^{**}$. We also give a characterization of the image of $L(X)$ which leads to an isomorphic copy of $L(X)$ from $K(X)$, without reference to the underlying Banach space.

2. The Arens products

The two Arens products are defined in stages according to the following rules. Let $\mathcal{A}$ be a Banach algebra. Let $A, B \in \mathcal{A}$; $f \in \mathcal{A}^*$; $F, G \in \mathcal{A}^{**}$.

DEFINITION 2.1. $(f *_1 A)B = f(AB)$. This defines $f *_1 A$ as an element of $\mathcal{A}^*$.

$(G *_1 f)A = G(f *_1 A)$. This defines $F *_1 f$ as an element of $\mathcal{A}^*$.

$(F *_1 G)f = F(G *_1 f)$. This defines $F *_1 G$ as an element of $\mathcal{A}^{**}$.

We will call $F *_1 G$ the first or m_1 product.

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