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PLANAR SURFACE IMMERSIONS

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Introduction

In this paper immersions of surfaces with boundary into the plane, $\mathbf{R}^2$, will be classified up to an equivalence relation called image homotopy. When two immersions are image homotopic, there is a smooth deformation through immersion images of one image to the other. This deformation may be drawn or visualized. It gives the appearance of a motion of the immersion in time. Before proceeding further, the reader might enjoy viewing the long image homotopy shown in Figure 3. This image homotopy is an example of mod-2 planar phenomena that we shall deal with in greater detail.

Planar surface immersions are a mixture of integral and mod-2 phenomena. For example, there are infinitely many image homotopy classes of immersions of a once punctured torus, but only *two* image homotopy classes for a surface of genus greater than one having a single boundary component. In the latter case, these two immersions are distinguished by a mod-2 quadratic form just as in [KB]. In fact, our results are quite similar to those of [KB], where immersions into the sphere, S^2, are classified up to image homotopy. Except for the use of quadratic forms, we do not assume familiarity with [KB]. The paper is organized as follows:

In Section 1 image homotopy is discussed and defined. Proposition 1.6 shows that $\mathcal{I}(N) \simeq \mathcal{R}(N)/\mathcal{M}(N)$ where $\mathcal{I}(N)$ denotes image homotopy classes of immersions of N, $\mathcal{R}$ denotes regular homotopy, and $\mathcal{M}(N)$ is the mapping class group of N (acting on $\mathcal{R}(N)$ by composition).

Section 2 discusses the role of curves on the surface. The Whitney–Graustein Theorem [W] is recalled and used to compute $\mathcal{R}(N)$. A boundary invariant, $B(f)$, of an immersion $f: N \to \mathbf{R}^2$ is defined in terms of the boundary curves of N. Proposition 2.3 computes $\mathcal{I}(N)$ for a k-holed disk in terms of the boundary invariant.

Section 3 discusses the generators of the mapping class group $\mathcal{M}(N)$ and then considers three important examples: (1) If $N = T$, a punctured torus, then $\mathcal{M}(N) \cong SL(2, \mathbf{Z})$ and $\mathcal{I}(N) \simeq \mathbf{Z}^+$. (2) If $N = T \# A$ (a torus with two holes), then the extra boundary component acts as a catalyst to reduce the toral part of the immersion modulo two. (3) If $N = T \# T$, a once-punctured double torus, then $\mathcal{I}(N)$ contains no more than two elements. These examples reflect the way particular sorts of diffeomorphisms of N act on

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