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ABELIAN SUBGROUPS OF Aut_k $(k[X, Y])$ AND APPLICATIONS TO ACTIONS ON THE AFFINE PLANE

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Introduction

In this study, we apply some theorems of group theory to study algebraic and non-algebraic actions of algebraic groups on the affine plane. The main object of study is the group Aut_k $(k[X, Y])$, which is denoted by $GA_2(k)$. When k is a field this group has a decomposition as an amalgamated free product $A *_B E$ of groups (§1). Since $GA_2(k)$ is (up to anti-isomorphism) the group of algebraic isomorphisms of the affine plane $\mathbf{A}^2(k)$, any action of a commutative algebraic group G on $\mathbf{A}^2(k)$ gives rise to a group homomorphism $G \to GA_2(k)$, the image of which homomorphism is then an abelian subgroup of $GA_2(k)$. Abelian subgroups of any amalgamated free product $A *_B E$ may be understood group theoretically, up to conjugacy, in terms of the groups A and E, and the containments $B \subset A$ and $B \subset E$, using certain results from combinatorial group theory, especially a theorem of Moldavanski (see 0.5). These essential facts are laid out in §0. By these means we are able to give a classification of any action of an algebraic group on the affine plane, up to equivalence (3.10 and 3.11).

The main theorem of §4 (Theorem 4.9) explicitly describes, up to equivalence, actions of the n-dimensional vector group G_a^n on the plane, as long as the field k is infinite. This generalizes the results of R. Rentschler and M. Miyanishi ([11] and [8]), which describe actions of G_a on the affine plane.

In Section 5, we employ these methods to give another proof of Gutwirth's theorem [5], which describes, up to equivalence, actions of the n-dimensional torus G_m^n on the plane. Again, we assume only that the field k is infinite. (Certain generalizations of this theorem involving faithful actions of tori on n-space can be found in [2] and [4].)

The writer is indebted to Professor Hyman Bass, who suggested these group theoretic methods as a means of describing actions of groups on the plane.

0. Some facts about subgroups of amalgamated free products

0.1. *Notation.* When G is a group and H is a subgroup of G, a right coset of G modulo H is an element of the coset space on which G acts on the right. Hence if $g \in G$, Hg is a right coset. If $h \in G$, we conjugate h by g by writing $h^g = g^{-1}hg$. Also, for $H \subset G$, we write H^g for $g^{-1}Hg$. We write

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