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ON CONVOLUTION SQUARES OF SINGULAR MEASURES

BY

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A classical result of Wiener and Wintner [6] asserts that there exists a singular probability measure μ on the circle group T such that $\hat{\mu}(n) = O(|n|^{-1/2+\varepsilon})$ as $n \to \infty$ for every $\varepsilon > 0$. Such a measure μ has the property that $\mu^2 = \mu * \mu$ is absolutely continuous and its Radon-Nikodym derivative with respect to Lebesgue measure belongs to $L^p(T)$ for all positive real numbers p (cf. [2] and [5]). In the present paper, we shall construct a singular probability measure μ, with support having zero Lebesgue measure, such that μ^2 has uniformly convergent Fourier-Stieltjes series.

Let λ be the normalized Lebesgue measure on T and let Z be the additive group of integers. We denote by $C_0(Z)$ the space of all functions on Z (i.e., two-sided sequences) that vanish at infinity. A mapping of $C_0(Z)$ into itself is called continuous if it is continuous with respect to the supremum norm of $C_0(Z)$. Our result can be stated as follows.

THEOREM. *Let K be a measurable subset of T having positive Lebesgue measure, and let ϕ be a continuous mapping of $C_0(Z)$ into itself. Then there exists a singular probability measure μ on T satisfying these conditions:*

(a) supp $\mu \subset K$ *and* $\lambda(\text{supp } \mu) = 0$;
(b) $\sum_{n=-\infty}^{\infty} |\hat{\mu}(n)^2 \cdot \phi(\hat{\mu})(n)| < \infty$;
(c) *The Fourier-Stieltjes series of μ^2 converges uniformly.*

In order to prove this theorem, we need some notation and lemmas. For $f \in C(T)$, we define

$$\|f\|_A = \sum_{n=-\infty}^{\infty} |\hat{f}(n)| \quad \text{and} \quad \|f\|_U = \sup_N \left\| \sum_{n=-N}^{N} \hat{f}(n)e^{int} \right\|_\infty .$$

Notice that the set of all $f \in C(T)$ with $\|f\|_A < \infty$ (or $\|f\|_U < \infty$) forms a Banach space (cf. [3]). Given $f \in L^1(T)$, let $f^{(2)} = f * f$ and let supp f denote the closed support of f. Throughout the following lemmas, we fix an arbitrary continuous mapping ϕ of $C_0(Z)$ into itself and write $\Psi(P) = P^2 \cdot \phi(P)$ for $P \in C_0(Z)$. We begin with improving Lemma 3.2 of [5] by applying Körner's idea in [4].

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