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ELEMENTARY AMENABLE GROUPS

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1. In order to explain the Hausdorff-Banach-Tarski paradox, von Neumann [19] introduced the class of amenable groups in 1929. Since then the theory of amenable groups has advanced in many fronts, for a survey see Day [4] and Greenleaf [9]. But algebraically the only known amenable groups and non-amenable groups were provided by von Neumann. He showed that finite groups and abelian groups are amenable and that the class of amenable groups, AG, is closed under four standard processes of constructing new groups from given ones: (I) *subgroups*, (II) *factor groups*, (III) *group extensions* and (IV) *direct unions*. As in Day [3] let EG be the smallest class of groups which contains all finite groups and all abelian groups and is closed under processes (I)–(IV). Then EG is contained in AG and, in fact, algebraically they constitute the only *known* amenable groups. We will call groups in EG *elementary amenable* groups. Von Neumann also showed that if a group contains a free subgroup on two generators then it is not amenable. Therefore, NF, the class of groups without free subgroup on two generators, contains AG. The notations AG and NF were also introduced by Day [3].

Von Neumann [19] asked whether $AG = NF$ and Day [3] pointed out it is not known whether $EG = AG$ or even $EG = NF$. We are unable to provide any new examples of amenable or non-amenable groups. But in this paper a better description of the known amenable groups will be given. More precisely, we will show that the groups in EG can be constructed from abelian groups and finite groups by applying processes (III) and (IV) only. By combining this fact with the existence of non-locally finite periodic groups, cf. [5], [20], we are able to conclude that $EG \subsetneq NF$. Therefore either $EG \neq AG$ or $AG \neq NF$ or both.

A finitely generated group G with a finite generating set F is said to be exponentially bounded if $(\text{card } F^n)^{1/n} \to 1$ as $n \to \infty$ where

$$F^n = \{x_1 \cdots x_n : x_i \in F\}.$$

This property is independent of the choice of F. Milnor [17] and Wolf [22] showed that if a finitely generated solvable group is exponentially bounded then it contains a nilpotent subgroup of finite index. By applying our description of elementary amenable groups we are able to extend their result to finitely

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