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A CONSISTENT FUBINI-TONELLI THEOREM FOR NONMEASURABLE FUNCTIONS

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The Fubini-Tonelli theorem asserts that if a nonnegative function of two real variables is measurable, then the iterated integrals exist and are equal (where infinite integrals are allowed). What if measurability is not assumed, but instead only that the iterated integrals exist?

It is not hard to see that this is a weaker hypothesis. For example, one can construct a permutation of **R** whose graph is nonmeasurable, and consider its characteristic function. The iterated integrals both exist and are 0.[2]

It is well known that using the continuum hypothesis, one can construct a nonnegative function of two real variables such that the iterated integrals exist but are not equal. The usual example is the characteristic function of the graph of a well ordering of **R**. Actually, this counterexample works under the milder hypothesis that every set of reals of power less than c is of measure 0.

Here we construct a model of ZFC in which there is no such counterexample. In other words, in this model, if the iterated integrals for a nonnegative function of two real variables exist, they are equal.

We fix M to be a countable transitive model of ZFC, and we use c to denote the Von Neumann cardinal of the continuum in the sense of M.

Within M, we can consider σ-algebra K of subsets of $2^{(c^+)}$ (the set of 0, 1-valued functions from c^+) which are generated by the sets $\{f: f(\alpha) = 0\}$, for ordinals $\alpha < c^+$. We consider K as equipped with the usual product measure which assigns each set $\{f: f(\alpha) = 0\}$ measure 1/2.

Our forcing conditions will be the elements of K of positive measure in M, and p extends q if and only if $p \subset q$. This notion of forcing is commonly referred to as "adding c^+ random reals".[3] In the usual way via Borel codes, there is a one-one correspondence between elements of K (which are in the ground model M) and subsets of actual $2^{(c^+)}$ (i.e., all functions from the c^+ of M into $\{0, 1\}$). In the standard way, one verifies that if G is a generic set of conditions, then $\bigcap G$ (defined using the one-one correspondence) consists of one element $\mathbf{f}$, and G is the set of all conditions which include $\mathbf{f}$. Such an $\mathbf{f}$ is said to be

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[2] We are grateful to Gerald Edgar for this example, and for conversations which lead to this paper.

[3] Random real forcing was first presented in [2]. In [3], uncountably many random reals are added to a ground model.