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POINTS OF SUPPORT FOR CLOSED CONVEX SETS

BY
A. J. LAZAR

A point x_0 of a closed convex subset K of a real Banach space X is called a point of support for K if there is a functional $x^* \in X^*$ such that $x^*(x_0) \leq x^*(x)$ for every $x \in K$ and $x^*(x_0) < x^*(x')$ for some $x' \in K$. S. Rolewicz proved in [4] that every separable K contains a point which is not a point of support for K and asked if every non-separable Banach space must contain a closed convex subset consisting only of points of support. He further asked what is the situation for $L^\infty[0, 1]$. We shall give below a partial answer to the first question and show that $L^\infty[0, 1]$ does indeed contain a subset with the required property.

All the Branch spaces considered are over the real field. The notation and the terminology are those of [1].

THEOREM 1. *A Banach space X whose dual is not weak* separable contains a closed convex subset consisting only of support points.*

Proof. For each countable ordinal α we shall construct, by transfinite induction, elements $x_\alpha \in X$ and functionals $x_\alpha^* \in X^*$ so that $x_\alpha^*(x_\alpha) = 1$ and $x_\alpha^*(x_\beta) = 0$ if $\beta \neq \alpha$. Choose $x_1 \in X$ and $x_1^* \in X^*$ such that $x_1^*(x_1) = 1$. Suppose that α is a countable ordinal and for each ordinal $\beta < \alpha$ we have chosen $x_\beta \in X$, $x_\beta^* \in X^*$ which satisfy $x_\beta^*(x_\beta) = 1$ and $x_\beta^*(x_\gamma) = 0$ if $1 \leq \gamma < \beta$ or $\beta < \gamma < \alpha$. Let

$$X_\alpha = \{x \in X : x_\beta^*(x) = 0 \text{ for every } \beta < \alpha\}.$$

We claim that X_α is a closed non-separable linear subspace of X. Indeed, if X_α were separable, there would be a sequence $\{y_n^*\}_{n=1}^\infty \subset X^*$ which is total over X_α. But then the linear span of $\{x_\beta^*\}_{\beta<\alpha} \cup \{y_n^*\}_{n=1}^\infty$ would be weak* dense in X^*, contrary to the hypothesis. Thus X_α is non-separable and we can choose $x_\alpha \in X_\alpha$ which is not in the closed linear span of $\{x_\beta : \beta < \alpha\}$. By the separation theorem there is $x_\alpha^* \in X^*$ such that $x_\alpha^*(x_\alpha) = 1$ and $x_\alpha^*(x_\beta) = 0$ for $\beta < \alpha$. Hence the existence of the families $\{x_\alpha\} \subset X$, $\{x_\alpha^*\} \subset X^*$ with the desired properties is proved.

Now, let

$$K = \overline{\text{conv}}\, \{x_\alpha : \alpha \text{ a countable ordinal}\}.$$

Clearly $K = \bigcup_\alpha \overline{\text{conv}}\, \{x_\beta : \beta < \alpha\}$ and for each $x \in \overline{\text{conv}}\, \{x_\beta : \beta < \alpha\}$ we can find

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