ILLINOIS JOURNAL OF MATHEMATICS
Volume 25, Number 4, Winter 1981

BOUNDED SOLUTIONS OF SCALAR, ALMOST PERIODIC LINEAR EQUATIONS

BY

RUSSELL JOHNSON

1. Introduction

Consider a scalar, non-homogeneous differential equation

$$(*) \qquad \dot{x} = a(t)x + b(t),$$

where a and b are Bohr-almost periodic functions. If the mean value of a is not zero, then Cameron [3] showed that (*) admits a unique bounded solution, and that said solution is almost periodic. One can now prove this result by observing that (*) has an exponential dichotomy, and appealing to general theorems (e.g., [5] or [13]). If a has mean value zero, and if $\int_0^t a(s)\, ds$ is bounded (and hence a.p. by Bohr's theorem), then it is easy to prove that one solution of (*) is bounded if and only if all solutions are almost periodic.

Our interest is in the case when a has mean value zero, but $\int_0^t a(s)\, ds$ is unbounded. The example of [12] (which uses that of [4]) shows that (*) may then admit bounded solutions, but *no* almost periodic solutions. Stating our results requires the introduction of the hull Ω of the function $f(t) = (a(t), b(t))$ (see 2.3). The space Ω may be given the structure of a compact, abelian topological group [14]. Let μ_0 be normalized Haar measure on Ω. Let Ω_β be the set of $\omega \in \Omega$ for which the equation $(*)_\omega$ defined by ω (see 3.1) admits a unique bounded solution. Let

$$\Omega_\alpha = \{\omega \in \Omega\colon \text{the equation } (*)_\omega \text{ admits an } \textit{almost automorphic} \text{ solution}\}$$

(almost automorphy generalizes almost periodicity; see 2.5 and [18]). It turns out that $\Omega_\beta \subset \Omega_\alpha$.

We prove the following: (i) Ω_α and Ω_β are residual subsets of Ω (3.10), and (ii) for "most" functions a, $\mu_0(\Omega_\beta) = 1$ (3.11). An example shows that (iii) $\mu_0(\Omega_\beta)$ may be zero (3.12). We also show that (iv) the example of [12] satisfies $\mu_0(\Omega_\alpha) = 1$ (3.14–3.16). Finally, (v) we indicate how altering the example of [12] might produce an example with $\mu_0(\Omega_\alpha) = 0$ (3.17). It should be noted that (i) is proved in the more general case when Ω is *minimal* (2.1).

We also consider the case when (*) admits *no* bounded solutions. Assuming that Ω is minimal, we show in 4.2 that residually many $\omega \in \Omega$ have the property

Received January 5, 1980.