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EXACT INTERVALS

BY
M. V. MIELKE

1. Introduction

In a previous paper we characterized those cosimplicial k-spaces $T\colon \Delta \to$ kTop whose left Kan extension $\mathrm{Lan}_R\, T$ along the right Yoneda functor $R\colon \Delta \to$ Simpl (Set) preserves finite products. It was shown that T arises from an "interval" T [1]. In this paper we extend these results by showing that $\mathrm{Lan}_R\, T$ is an exact functor (preserves finite limits and colimits) if and only if the reflection of T [1] into the category of T_0 spaces is a Hausdorff space. In the classical case, where T is the cosimplicial space of affine simplexes, T [1] is the standard unit interval I and $\mathrm{Lan}_R\, T$ is the geometric realization functor.

2. Preliminaries

Recall, from [4], that the category Int of Intervals has, as objects, the non-empty, linearly ordered, bounded sets X equipped with a connected compactly generated topology (in the sense of [5]) for which X^n, the n-fold product in kTop, has the weak topology relative to the family $\{gX_n\}$, $g \in S(n)$, the permutation group on n objects, where

$$X_n = \{(x_1, \ldots, x_n) \mid x_1 \le \cdots \le x_n\} \subset X^n$$

and

$$gX_n = \{(x_{g1}, \ldots, x_{gn}) \mid (x_1, \ldots, x_n) \in X_n\} \subset X^n,$$

and has, as morphisms, the continuous, non-decreasing, endpoint preserving maps. Theorem 4.1 of [4] shows that the correspondence $X \to T_X\colon \Delta \to$ kTop, where $T_X[n] = X_n$, defines an equivalence between Int and the full subcategory of cosimplicial k-spaces determined by those $T\colon \Delta \to$ kTop for which T [1] is nonempty and connected, and $\mathrm{Lan}_R\, T$ preserves finite products. The aim of this paper is to characterize the category EInt of exact intervals, i.e. to explicitly describe those $X \in$ Int for which $\mathrm{Lan}_R\, T_X$ is exact. Note that $\mathrm{Lan}_R\, T_X$ is exact if and only if it preserves equalizers since, in general, $\mathrm{Lan}_R\, T$ is cocontinuous (it is left adjoint to the singular functor $X \to$ Set (T, X)) and the preservation of finite limits is equivalent to the preservation of finite products and equalizers [2, Section 2, p. 108].

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