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EXTREME INVARIANT EXTENSIONS OF PROBABILITY MEASURES AND PROBABILITY CONTENTS

BY
HARALD LUSCHGY

1. Introduction

Let G be a semigroup which acts from the left on a set X and let $\mathscr{A}$ and $\mathscr{B}$ be invariant σ-algebras on X with $\mathscr{B} \subset \mathscr{A}$. In this paper we characterize the extreme points of the convex set of all invariant probability measures on $\mathscr{A}$ which extend a given probability measure P on $\mathscr{B}$ and we give an extremal integral representation in this set. This problem has been investigated by Farrell [8] and by several other authors for $\mathscr{B} = \{\emptyset, X\}$ and by Plachky [17] for $G = \{\mathrm{id}_X\}$.

Starting with a known characterization by an approximation property [14] we clarify its relation to the notion of pairwise sufficient σ-subalgebras of $\mathscr{A}$. For a wide class of measurable spaces $(X, \mathscr{A})$ and semigroups G the extreme invariant extensions of P turn out to be those invariant extensions whose conditional probabilities with respect to the σ-algebra of P-almost invariant $\mathscr{B}$-measurable sets are multiplicative modulo an averaging process. As an application of a Choquet type theorem of v. Weizsäcker and Winkler [20] we obtain an extremal integral representation in the set of invariant extensions of P.

Finally, given invariant algebras $\mathscr{A}$ and $\mathscr{B}$ with $\mathscr{B} \subset \mathscr{A}$ we derive characterizations of the extreme points of the convex set of all invariant probability contents on $\mathscr{A}$ which extend a given probability content on $\mathscr{B}$.

2. Preliminaries

Let X be a set, let G be a semigroup which acts from the left on X, and let $\mathscr{A}$ be an invariant algebra on X, i.e.

$$g^{-1}A = \{x \in X: gx \in A\} \in \mathscr{A} \quad \text{for all } g \in G,\ A \in \mathscr{A}.$$

An additive set function $\mu: \mathscr{A} \to \mathbf{R}$ is called invariant if $\mu(g^{-1}A) = \mu(A)$ for all $g \in G$, $A \in \mathscr{A}$. By $ba(\mathscr{A})$ we denote the space of all bounded, (finitely) additive real set functions on $\mathscr{A}$ and by $ba(\mathscr{A})_G$ we denote the subspace of all invariant elements. Then $ba(\mathscr{A})_G$ is an order complete Banach sublattice of $ba(\mathscr{A})$. We may identify $ba(\mathscr{A})$ with the topological dual $B(\mathscr{A})'$ of $B(\mathscr{A})$, where $B(\mathscr{A})$ denotes the closed linear hull of the set $\{1_A: A \in \mathscr{A}\}$ in the Banach lattice $B(X)$

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