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FIBERING COMPLEX MANIFOLDS OVER S^3 AND S^4

BY

STEVEN M. KAHN[1]

1. Introduction

For any closed manifold N, there are obstructions to the smooth fibering over N, of a stably almost complex manifold M. This paper examines those obstructions which are given by the Stiefel-Whitney, Pontrjagin and Chern numbers of M.

A complete solution, for a given base space N, consists of finding a set of obstructions whose vanishing on a manifold M guarantees the existence of a fibration over N, with total space complex cobordant to M. Thus, we are trying to find those cobordism classes $\omega \in \Omega_*^u$ which contain a representative fibered over N. If such a representative exists, we say that the class ω fibers over N. Note that for fixed N, the set of ω which fiber over N is an ideal in Ω_*^u, which we denote by $\mathrm{Fib}_{\Omega_*^u}(N)$.

It was shown in [4, p. 68] that a class $\omega \in \Omega_*^u$ fibers over S^1 if and only if the signature $\sigma(\omega)$ is zero. Nelson [8, Theorem 3.8] proved that this signature condition is also the only obstruction to fibering a unitary class over S^2, while the general result for any connected surface B^2, is the following (see [1]):

A class $\omega \in \Omega_n^u n \geq 2$, fibers over B^2 if and only if

$$\sigma(\omega)\begin{cases} = 0 & \text{if } \chi(B) \geq 0 \\ \equiv 0 \pmod 4 & \text{if } \chi(B) < 0, \end{cases}$$

where χ denotes the Euler characteristic.

For fiberings over a manifold N of dimension greater than two, the situation is quite different. The necessary signature conditions in this case are in general, no longer sufficient. There is an interplay that surfaces here, between fiber and total space cohomologies, involving actions of both stable and unstable operations on characteristic classes. The result is a more involved set of obstructions.

This paper completely determines the obstructions to fibering a class $\omega \in \Omega_*^u$ over S^3 and S^4 (the two solutions are precisely the same). As consequences we obtain results for some other 3 and 4-manifolds. We also get a partial description of the fibering ideal of $S^2 \times S^2$ and offer a conjecture as to what the complete answer should be. In addition, since almost all the fibrations we construct over the spheres actually fiber over CP(3), and since all obstruc-

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