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A BOUND ON THE RANK OF $\pi_q(S^n)$

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Let p be an odd prime. In [7], Toda constructs two fibrations which he uses to give a bound on the exponent of $\pi_q(S^n)$. We show here how these fibrations can be used to give a bound on the rank of $\pi_q(S^n)$. These groups are known to be finitely generated from the work of Serre [6]. It suffices to consider odd n since, again from Serre [5],

$$\pi_q(S^{2n})_{(p)} \cong \pi_{q-1}(S^{2n-1})_{(p)} \oplus \pi_q(S^{4n-1})_{(p)}.$$

Although analogues of Toda's fibrations exist for the prime 2, (James [2]), the arguments given here fail because unlike the situation for odd primes, $\pi_*(X; Z/2Z)$ fails to be a $Z/2Z$ vector space.

Let X be a compactly generated topological space with basepoint "$*$". Let $J_k(X)$ denote the kth stage of the James Construction on X. That is, $J_k(X) = X^k/\sim$ where

$$(x_1, \ldots, x_{j-1}, *, x_{j+1}, \ldots, x_k) \sim (x_1, \ldots, x_{j-1}, x_{j+1}, *, x_{j+2}, \ldots, x_k).$$

After localizing to p, there are fibrations up to homotopy:

$$J_{p-1}(S^{2n}) \xrightarrow{i} \Omega S^{2n+1} \xrightarrow{H} \Omega S^{2pn+1}$$

and

$$S^{2n-1} \xrightarrow{j} \Omega J_{p-1}(S^{2n}) \xrightarrow{T} \Omega S^{2pn-1}.$$

We use mod-p homotopy in order to be able to make use of vector space arguments. Of course,

$$\dim \left(\pi_q\left(S^{2n+1}; \frac{Z}{pZ}\right)\right) = \text{rank}\,(\pi_q(S^{2n+1})) + \text{rank}(\pi_{q-1}(S^{2n+1})) \quad \text{for } q > 2n+2.$$

Let x be a non-zero element of $\pi_q(S^{2n+1}; Z/pZ)$ with $n > 0$. Construct a non-zero element x', in some mod-p homotopy group of some sphere as follows:

If $H_\#(x) \neq 0$, let $x' = H_\#(x)$. If $H_\#(x) = 0$, select a non-zero element,

$$y \in \pi_{q-1}(J_{p-1}(S^{2n})) = \pi_{q-2}(\Omega J_{p-1}(S^{2n})),$$

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