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REDUCED Σ-SPACES

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Σ-spaces and reduced Σ-spaces were first introduced by O. Loos [10] in 1972 as a generalisation of reflection spaces and symmetric spaces. Loos proved that if Σ is a compact Lie group then any Σ-space is a fibre bundle over a reduced Σ-space which, in turn, is homogeneous. This raises several questions about such spaces when the compactness assumption is removed. Our purpose in this paper is to introduce and study other classes of Σ-spaces, essentially for the reduced case. We hope that the terminology introduced here will prove to be acceptable and so lead to some standardisation of language within the subject.

Basic properties of any reduced Σ-space M (defined by (Σ1)–(Σ6)) are given in Section 1. Most of these are contained in [10] but, for completeness, we have selected what is needed for our purpose and given a self-contained account. Thus, properties of the group G_M are due to Loos, as are Lemma 1·6 and Theorem 1·7 which are fundamental. We have introduced tensor fields S^σ as a natural extension of the tensor field S defined on s-manifolds [2].

If M is a reduced Σ-space and Σ is compact then, by [10], G_M is a Lie transformation group of M on which Σ acts by automorphisms, and the associated coset space is a reduced Σ-space isomorphic to M. An important step in the proof is to show that M admits a particular affine connection. This connection can be characterised by two properties. In Section 2 we select one of these, (Σ7), to define a reduced affine Σ-space, and the space is then called canonical if the second property, (Σ8), holds. We consider such spaces with Σ possibly non-compact; in particular, Theorem 2·7 gives a coset space presentation for the canonical case. It follows, for the same case, that the group G_M is a connected Lie transformation group of M; this result is basic for later applications.

In Section 3 we study the case when Σ is cyclic, and show that, as for Σ-compact, such a space always admits the canonical connection. Furthermore, it is then, essentially, just an affine s-manifold. Other versions of these results can be found in [7] and [13]. It should be noted, however, that the compact and cyclic cases differ in their coset space presentation and we give some examples in that direction.

Finally, Section 4 deals with reduced Riemannian Σ-spaces and, in Theorem 4·4, we obtain a coset space presentation which shows that the existence of an invariant metric implies the existence of the canonical connection. Then by Corollary 4·6 we see that Σ can always be compactified so as to extend its

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