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POLYNOMIAL GROUP LAWS OVER VALUATION RINGS

BY

WILLIAM C. WATERHOUSE[1]

Let A be a discrete valuation ring with fraction field K and residue field k. Let R be a finitely generated flat A-algebra, and suppose that $R \otimes K$ and $R \otimes k$ are polynomial rings. Must R be a polynomial ring? Proofs of this have been given only for one variable (Danilov, unpublished; Kambayashi-Miyanishi [5]) and for two variables if k is algebraically closed of characteristic zero (Kambayashi [4]). The situation is better, however, when R is the ring of functions $A[G]$ on an affine group scheme G. This was indeed the context in which Weisfeiler and Dolgachev [7] first raised the question, since, when char $(k) = 0$, the result for $A[G]$ is easily established by Lie theory. They were able to establish it also when char $(K) = p$ and k is perfect and the generic fiber G_K is G_a^n. The theorem was later proved [8] for all commutative G. In this paper it is proved for group schemes without restriction:

THEOREM. *Let G be a flat affine group scheme of finite type over a discrete valuation ring A. Assume the two fibers are represented by polynomial rings. Then $A[G]$ is a polynomial ring.*

As in [8] and [4], the proof is in outline an induction using Néron blow-ups. Some new results on the structure of polynomial groups over fields are needed for the argument and will be established first.

1. Review of Néron blow-ups

Let $G = \operatorname{Spec} A[G]$ be a flat affine scheme of finite type over the discrete valuation ring A. Tensoring with the fraction field, we can by flatness identify $A[G]$ with a subalgebra of $K[G] = A[G] \otimes_A K$. Let X be a closed subscheme of the special fiber G_k, so X is defined by some ideal $J = (\pi, f_1, \ldots, f_n)$, where π is the uniformizer. The subalgebra

$$A[\pi^{-1}J] = A[G][\pi^{-1}f_1, \ldots, \pi^{-1}f_n]$$

represents a scheme G^X which one says is obtained by *blowing up* X in G. We will need the following properties, of which (b) is the crucial one (cf. [9, Theorem 1.4]).

(a) Let G' be any other such flat affine scheme. Any map $G' \to G$ sending G'_k into X factors through G^X.

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