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LAPLACIANS AND RIEMANNIAN SUBMERSIONS WITH TOTALLY GEODESIC FIBRES

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0. Introduction

0.0 The Laplace-Beltrami operator on a compact Riemannian manifold (M, g) can be viewed as a natural generalization of the ordinary Laplacian on a bounded domain in $\mathbf{R}^n$. In particular, most properties of eigenvalues and eigenfunctions of the latter with either Dirichlet or Neuman boundary values carry out to the spectral data of the former.

As an example, the famous Faber-Krahn inequality (see [10], page 188) which bounds from below the product of the area of a domain in $\mathbf{R}^2$ by its first eigenvalue finds its counterpart in J. Hersch's theorem (cf. [14]) which gives an upper bound of the product of the first nonzero eigenvalue of any Riemannian metric on the 2-sphere S^2 by the Riemannian area. The Faber-Krahn inequality generalizes to domains in $\mathbf{R}^n$. In [2], M. Berger shows that such an extension does not exist for the n-sphere S^n, if one insists on the upper bound being sharp for the standard metric. This still leaves some hope for the existence of an upper bound.

Analogously, Courant's nodal line theorem (see [10], page 452) according to which the ith eigenfunction has at most i nodal domains (i.e., connected components of the complement of its zero set) is valid for a general Riemannian manifold. Such a result suggests that eigenfunctions corresponding to high eigenvalues should be complicated. In particular they were expected to have more than two nodal domains.

In the same vein, it is reasonable to believe that the more symmetries a Riemannian metric has, the more multiple its eigenvalues are. In particular, the standard metric on the sphere was expected to be the metric with the largest multiplicity of its first nonzero eigenvalue.

All these guesses turn out to be wrong for appropriate choices of (M, g). The right choices all belong to the class of manifolds to which this article is devoted, namely, *Riemannian submersions with totally geodesic fibres*. These metrics are the next simple metrics after Riemannian products. This probably indicates that, in a sense, Riemannian metrics on compact manifolds form a wider family than domains in $\mathbf{R}^n$.

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