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ON EXTREME POINTS

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This note contains a proof of the following:

THEOREM. *Let E be a non-reflexive real Banach space. There exist closed bounded convex sets A, C_1, C_2 in E with the following properties:*

(a) *The point 0 is an exposed point of A.*

(b) *The point 0 is not an extreme point of B, the weak* closure of A in the second dual E^{**}. If E is not weakly sequentially complete, 0 is in fact the average of two exposed points of B.*

(c) *The point 0 is not in the convex hull of $C_1 \cup C_2$, but it is an exposed point of the closed convex hull of $C_1 \cup C_2$.*

Recall [1, V.1. (8)] that a point x of a convex set A is *exposed* if there is a continuous linear functional f such that $f(x) < f(y)$ for all $y \in A$, $y \neq x$. In such a case we say that f (or $-f$) *exposes* $x \in A$.

Proof. Case 1. Suppose that E is not weakly sequentially complete. Then there is a sequence $\{z_n\}$ in E which is weak* convergent in E^{**} to an element $\tilde{x}$ not in E. We choose now two linear functionals $g, h \in E^*$ as follows: first, $g \neq 0$ and $g(\tilde{x}) = 0$; pick $a \in E$ such that $g(a) = 1$ and choose h such that $h(\tilde{x}) = 1$, $h(a) = 0$.

Observe that $h(z_n) \rightarrow h(\tilde{x}) = 1$ and therefore by ignoring a finite number of terms we can (and will) assume that $h(z_n) \geq \frac{1}{2}$ for all $n \geq 1$.

Define

$$\alpha_n = |g(z_n)| + 1/n$$

$$\beta_n = (h(z_n) + 1/n)^{-1}$$

$$x_n = \beta_n(z_n + \alpha_n a)$$

$$y_n = \beta_n(-z_n + \alpha_n a).$$

It is easy to see that for each $n \geq 1$,

$$g(x_n) > 0, \quad g(y_n > 0), \tag{1}$$

$$\tfrac{1}{3} \leq h(x_n) < 1, \quad -1 < h(y_n) < -\tfrac{1}{3}, \tag{2}$$

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