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A STRONGER FORM OF THE BOREL-CANTELLI LEMMA

BY

THEODORE P. HILL[1]

1. Introduction

Let $A_1, A_2, \ldots$ be a sequence of measurable sets in a probability space (X, A, P), let $p_1 = P(A_1) > 0$, and, for $n > 1$, let p_n be the conditional probability of A_n given F_{n-1} (the σ-field generated by $A_1, \ldots, A_{n-1}$). Let $\chi(A)$ denote the indicator function of the set A, and, for $n \geqslant 1$, let

$$S_n = \sum_{j=1}^{n} \chi(A_j) \quad \text{and} \quad s_n = \sum_{j=1}^{n} p_j.$$

Let $\log_k r$ denote the *k-th* iterated logarithm of r (for example, $\log_3 r = \log\log\log r$). The main objective of this paper is to prove:

THEOREM 1. *For any positive integer k, both*

(1)
$$\lim_{n\to\infty} (S_n - s_n)/[s_n \log_1(s_n) \cdots \log_{k-1}(s_n) \log_k^2(s_n)]^{1/2} = 0$$

and

(2)
$$\lim_{n\to\infty} (S_n - s_n)/[S_n \log_1(S_n) \cdots \log_{k-1}(S_n) \log_k^2(S_n)]^{1/2} = 0$$

a.s. on the set where $\Sigma_1^\infty p_j = \infty$.

Theorem 1 brings the classical Borel-Cantelli lemma much closer to the central limit theorem and law of the iterated logarithm, without any additional assumptions concerning the divergence of the sums of the variances of the random variables in question, assumptions quite essential in both latter results. It sharpens Levy's conditional form of the Borel-Cantelli lemma [5, Corollary 68, p. 249], and an improved version due to Dubins and Freedman ([2, Theorem 1] or [6, Corollary VII-2-6, p. 152]) which is stated

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