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MULTI-DIMENSIONAL VOLUMES, SUPER-REFLEXIVITY AND NORMAL STRUCTURE IN BANACH SPACES[1]

BY

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1. Introduction

The notion of the n-dimensional volume enclosed by $n + 1$ vectors in a Banach space, E, was introduced by Silverman [12], and some of the connections between higher dimensional volumes and geometric properties of E were studied in [13] and [6]. In particular, it was shown that a k-uniformly rotund Banach space is super-reflexive and has normal structure. (Definitions are given below.) The present paper is a more detailed study of the relationships between enclosed volumes, super-reflexivity and normal structure of Banach spaces.

James proved in [9] that if E is not super-reflexive, then for every $\delta > 0$ there are $\{x_1, x_2\} \in B$, the unit ball of E, such that

$$\left\|\frac{x_1 + x_2}{2}\right\| \geqslant 1 - \delta$$

while $A(x_1, x_2) = \|x_1 - x_2\| \geqslant 2 - \delta$. A consequence of Theorem 3.1 of [6] is that if E is not super-reflexive, then for every integer $k > 0$ there are vectors $\{x_1, x_2, \ldots, x_k\} \subset B$ such that

$$\left\|\frac{x_1 + x_2 + \cdots + x_k}{k}\right\| \geqslant 1 - \delta$$

with $A(x_1, x_2, \ldots, x_k) > 0$. In section 3 we generalize these results and show that if E is not super-reflexive, then, for every integer $n > 0$, there are vectors $\{x_1, x_2, \ldots, x_{n+1}\} \subset B$ such that

$$\left\|\frac{x_1 + \cdots + x_{n+1}}{n + 1}\right\| \geqslant 1 - \delta$$

while $A(x_1, x_2, \ldots, x_{n+1}) \geqslant 2^n - \delta$. This should be contrasted with the situation for l_2, where $A(x_1, x_2, \ldots, x_{n+1}) \geqslant \varepsilon$ implies that

$$\left\|\frac{x_1 + \cdots + x_{n+1}}{n + 1}\right\| \leqslant \left[1 - \frac{n}{n + 1}\left(\frac{\varepsilon^{2/n}}{(n + 1)^{1/n}}\right)\right]^{1/2}.$$

[1] Some of the results of this paper are contained in the Ph. D. dissertation of the first author.

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