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COARSE TOPOLOGIES IN NONSTANDARD EXTENSIONS VIA SEPARATIVE ULTRAFILTERS

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0. Introduction

Let $^*\mathcal{M}$ be a nonstandard extension (ω_1-saturated will do) of a suitably large ground model $\mathcal{M}$. If $A \in \mathcal{M}$ then $^*A$ will denote the image of A in $^*\mathcal{M}$ under the canonical embedding, and $^*[A]$ will denote the set $\{^*a : a \in A\}$. If $\langle X, \tau \rangle$ is a topological space in $\mathcal{M}$ then $^*[\tau]$ is in general no longer a topology but is a basis for what we call the *coarse topology* on $^*X$. This is one of two natural topologies one could put on $^*X$ (the other, generated by $^*\tau$, is the "Q-topology" (see [1], [2], [4], [7], [8]) and is much finer) and is closely related to the "S-topology" (see [6], [8]) used in monad constructions in the setting of uniform spaces.

Our interest here is centered on the question of when $^*X$ (always with the coarse topology) enjoys some of the usual separation properties. As an example, if $\langle \mathbf{R}, v \rangle$ denotes the real line with its usual topology then $^*\mathbf{R}$ can never be a T_0-space when $^*\mathcal{M}$ is ω_1-saturated. In fact, if $^*\mathcal{M}$ is an enlargement (e.g. $^*\mathcal{M}$ is $|\mathcal{M}|^+$-saturated) then $^*X$ is never T_0 for infinite X.

As far as we know, it is an open question whether $^*X$ can be T_0 when X is infinite and $^*\mathcal{M}$ is ω_1-saturated. However, with the help of extra set theory (notably Martin's Axiom (MA) and the Continuum Hypothesis (CH)), we can construct extensions $^*\mathcal{M}$ in which $^*X$ can be T_0 (even Tichonov) for a large class of spaces X.

To begin with, we confine our attention to ultrapower extensions $^*\mathcal{M} = \Pi_D(\mathcal{M})$ where D is a free (that is, nonprincipal) ultrafilter on a countable set I. Then $^*\mathcal{M}$ is automatically ω_1-saturated (since D is countably incomplete) and its elements are equivalence classes $[f] = [f]_D$ of functions $f \in {}^I\mathcal{M}$;

$$[f] = \{g \in {}^I\mathcal{M} : \{i \in I : g(i) = f(i)\} \in D\}.$$

In the case of the nonstandard real line, for example, we have $[f] \; ^*\!< [g]$ iff $\{i \in I : f(i) < g(i)\} \in D$. For topological spaces $\langle X, \tau \rangle$, if $U \in \tau$ then $^*U = \{[f] : \{i : f(i) \in U\} \in D\}$. (Note that X and $^*[X]$ are naturally homeomorphic ($x \mapsto {}^*x$ is a homeomorphism) and that $^*[X] \subseteq {}^*X$ is a dense subset. This is true for any extension $^*\mathcal{M}$.)

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