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ON GENERAL LATTICE REPLETENESS AND COMPLETENESS

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Introduction

In this paper we wish to initiate a systematic study of various concepts pertaining to repleteness or completeness of a lattice. Special cases include such notions as realcompactness, α-completeness, Borel completeness, N-compactness, almost-realcompactness, and so on.

Specifically we consider an arbitrary set X and an arbitrary lattice $\mathscr{L}$ of subsets of X. We denote the algebra of subsets of X generated by $\mathscr{L}$ by $\mathscr{A}(\mathscr{L})$ and the set of all (finitely additive) two-valued measures on $\mathscr{A}(\mathscr{L})$ by $I(\mathscr{L})$. We then consider various subsets of $I(\mathscr{L})$ and, denoting the general element of $I(\mathscr{L})$ by μ, we demand that the support of μ, $S(\mu)$ be non-empty for μ in these subsets. Particular choices of these subsets, in the case where X is a topological space and $\mathscr{L}$ a particular lattice of subsets of X, give all the special cases referred to above as well as many others.

We proceed to analyze in the abstract setting of $\langle X, \mathscr{L}\rangle$ interrelations between these various concepts of repleteness-completeness, and then consider the important situation of two lattices $\mathscr{L}_1$, $\mathscr{L}_2$, with $\mathscr{L}_1 \subset \mathscr{L}_2$, and investigate when $\mathscr{L}_1$-(repleteness-completeness) implies $\mathscr{L}_2$-(repleteness-completeness), and conversely. Our results subsume all the known relationships in the special cases referred to above and they also yield new applications. We give a few of the applications, but it should be clear from these that many more such applications are available by appropriate choice of the lattices.

We then investigate particular lattices in subsets of $I(\mathscr{L})$ which are replete-complete and show how these can be utilized in constructing repletions-completions in particular cases.

The important point throughout the paper is that we can systematically treat all cases of repleteness-completeness, uniformly, by general measure-theoretic techniques. This was done to a limited extent in [2], just for repleteness, and in [10] using filter arguments with just one lattice and with just certain completeness notions. The advantage of the measure approach is that it is particularly simple with respect to extension-restriction matters and that much of it can be extended to the case of arbitrary measures—not necessarily two-valued; in this paper we will just pursue the case of

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