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INDEX OF HECKE OPERATORS

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1. Introduction

Let M be a complete Riemannian manifold. Suppose that the discrete group Γ acts isometrically and properly discontinuously on M with compact quotient $\overline{M} = \Gamma \setminus M$. Even though $\overline{M}$ need not be a manifold, the heat equation method [5] may be used to study the spectral theory of $\overline{M}$. Suppose S is a set of isometries satisfying (2.1). Associated to S is the Hecke operator T_S, and we will study the asymptotic behavior of its trace on the eigenspaces of the Laplacian.

Now suppose in addition that M is oriented and that Γ and S are orientation preserving. One may define the signature complex of $\overline{M}$ and consider the signature, $\mathrm{Sign}(T_S)$, of the Hecke operator T_S. We will give an explicit formula for $\mathrm{Sign}(T_S)$ in Theorem 4.1. Our approach is a natural extension of the technique used in [8] to prove the equivariant signature theorem. Since M need not be compact, it appears that the original proof of the equivariant signature theorem by Atiyah and Singer [2] does not generalize to compute $\mathrm{Sign}(T_S)$. In particular, Atiyah and Singer relied upon the representation theory of compact groups.

If $M = G/K$ is a globally symmetric space, then the Hecke operators associated to certain sets S of isometries have been studied by several authors [9], [11], [12], [13]. The most effective technique has been the Selberg trace formula. Of course, the trace formula can only be used when M admits a transitive group of isometries, so our results are more general. Furthermore, even in the case of symmetric spaces, Selberg's work does not immediately give an explicit formula for $\mathrm{Sign}(T_S)$. To derive our results from the Selberg trace formula, certain complicated orbital integrals must be simplified. Apparently, this has not been carried out except in special cases. On the other hand, the trace formula can give an expression for the individual traces of T_S on each harmonic piece of the signature complex, $\mathrm{Tr}_+(T_S)$ and $\mathrm{Tr}_-(T_S)$, rather than just the difference

$$\mathrm{Sign}(T_S) = \mathrm{Tr}_+(T_S) - \mathrm{Tr}_-(T_S).$$

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