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INNER PRODUCTS ON A GREEN RING FOR FINITE GROUPS WITH A CYCLIC *P*-SYLOW SUBGROUP

BY

I. REITEN AND K.W. ROGGENKAMP

Introduction

Let G be a finite group with a cyclic p-Sylow subgroup and let R be an unramified extension of the p-adic integers, for some prime number p. Denote by $\mathfrak{p}$ the radical of R and by K its field of quotients. Then L will be either R or $R/\mathfrak{p} = k$. In addition we assume k to be a splitting field for G. (This is a technical assumption which is only used in Lemma 1.2 to guarantee that the projectives in a minimal projective resolution of R over RG are indecomposable. It is superfluous when $L = k$ (see [10]), and if the p-Sylow subgroup has order p [6], [14].) Let ${}_{LG}M^0$ be the category of L-free finitely generated left LG-modules, and $\mathfrak{A}_L(G)$ the Green ring of the LG-modules in ${}_{LG}M^0$, that is, the elements in $\mathfrak{A}_L(G)$ are generated by the isomorphism classes of modules in ${}_{LG}M^0$. Addition is induced from the direct sum and multiplication from the tensor product over L. We often do not distinguish carefully between the modules in ${}_{LG}M^0$ and the objects in $\mathfrak{A}_L(G)$.

Denote by L_0 the trivial LG-module, and consider

$$\mathscr{P}_{L_0}\colon \cdots \to Q_i \to Q_{i-1} \to \cdots \to Q_0 \to L_0 \to 0,$$

a minimal projective resolution of L_0. We note that if $L = R$ and the p-Sylow subgroup of G has order p, then all nonprojective indecomposable R-free RG-modules in the principal block occur as syzygies in $\mathscr{P}_{R_0}$ [6], [14]. Let $\mathfrak{A}_L^0(G)$ be the subring of $\mathfrak{A}_L(G)$ generated by the finitely generated projective LG-modules and the syzygies in $\mathscr{P}_{L_0}$. If Ω_i is such a syzygy, then $\mathscr{P}_{L_0} \otimes_L \Omega_i$ gives a projective resolution of Ω_i, so that $\Omega_j \otimes_L \Omega_i$ decomposes into a direct sum of a projective and a syzygy module of Ω_i, which is also a syzygy of L_0. $\mathfrak{A}_L^1(G)$ denotes the ideal in $\mathfrak{A}_L^0(G)$ generated by the finitely generated projective modules.

In this note we study a bilinear form $[\ \ ,\ \ ]$ on $\mathfrak{A}_L^0(G)$, and we show that this form is nondegenerate unless $L = R$ and the p-Sylow subgroup of G has order 2. To prove this, denoting by Q the rational numbers, we consider the associated ring $\tilde{\mathfrak{A}}_L^0(G) = Q \otimes_Z \mathfrak{A}_L^0(G)$ with corresponding ideal $\tilde{\mathfrak{A}}_L^1(G)$ and

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