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AN APPROACH TO NUMERICAL RANGES WITHOUT BANACH ALGEBRA THEORY

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1. Introduction

The theory of numerical ranges in unital Banach algebras has been extensively developed in recent years. Its most remarkable results can be found in [6] and [7]. Numerical range techniques have been successfully applied to nonassociative normed algebras (for example, see [5], [15], [11], [19], [20]).

Since the concept of numerical range of an element of an (even nonassociative) unital Banach algebra does not depend on the product of the algebra but only on the underlying Banach space and on the unit, one is tempted to consider numerical ranges in an arbitrary Banach space X in which a norm-one element u has been selected. (We shall say that the pair (X, u) or simply X is a *numerical range space*). As in unital Banach algebras we can consider the *state space*

$$D(X) = \{ f \in X' : \|f\| = f(u) = 1 \}$$

and the *numerical range* of an element x in X, namely

$$V(x) = \{ f(x) : f \in D(X) \}.$$

This idea appears implicitly in the classical paper by Bohnenblust and Karlin [4] and has been shown to be useful in obtaining relevant results on numerical ranges in certain Banach algebras (for example, see [8]).

In this paper we begin a methodical consideration of numerical range spaces. A number of results on numerical ranges in unital Banach algebras can be not only extended to our general context but even improved in their original context. This is the case for example with our Corollary 2.9 which improves Theorem 2.4 of [17] and whose proof uses essentially numerical ranges in Banach spaces which need not be Banach algebras.

The leitmotiv in our work is the problem considered by R.R. Smith in [17] and [18] in the context of unital complex Banach algebras. This problem can be posed as follows. Given an element F in the second dual A'' of a unital

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