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VERDIER AND STRICT THOM STRATIFICATIONS IN O-MINIMAL STRUCTURES

TA LÊ LOI

0. Introduction

0.1 DEFINITION. An *o-minimal structure* on the real field $(\mathbf{R}, +, \cdot)$ is a family $\mathcal{D} = (\mathcal{D}_n)_{n\in\mathbf{N}}$ such that for each $n \in \mathbf{N}$:

(1) $\mathcal{D}_n$ is a boolean algebra of subsets of $\mathbf{R}^n$.
(2) If $A \in \mathcal{D}_n$, then $A \times \mathbf{R}$ and $\mathbf{R} \times A$ belong to $\mathcal{D}_{n+1}$.
(3) If $A \in \mathcal{D}_{n+1}$, then $\pi(A) \in \mathcal{D}_n$, where $\pi\colon \mathbf{R}^{n+1} \longrightarrow \mathbf{R}^n$ is the projection on the first n coordinates.
(4) $\mathcal{D}_n$ contains $\{x \in \mathbf{R}^n\colon P(x) = 0\}$ for every polynomial $P \in \mathbf{R}[X_1, \ldots, X_n]$.
(5) Each set belonging to $\mathcal{D}_1$ is a finite union of intervals and points. (o-minimality)

A set belonging to $\mathcal{D}$ is called *definable* (in this structure). *Definable maps* are maps whose graphs belong to $\mathcal{D}$.

Many results in Semialgebraic Geometry and Subanalytic Geometry hold true for o-minimal structures on the real field. Recently, o-minimality of many interesting structures on $(\mathbf{R}, +, \cdot)$ has been established, for example, structures generated by the exponential function [W1](see also [LR] and [DM1]), real power functions [M2], Pfaffian functions [W2] or functions defined by multisummable powerseries [DS]. For more details on o-minimal structures we refer the readers to [D] and [DM2] (compare with [S]).

We now outline the main results of this paper. Let $\mathcal{D}$ be an o-minimal structure on $(\mathbf{R}, +, \cdot)$. In Section 1, we prove that the definable sets of $\mathcal{D}$ admit Verdier Stratification. We also show that the Verdier condition (w) implies the Whitney condition (b) in $\mathcal{D}$. Note that the theorems were proved for subanalytic sets in [V] and [LSW] (see also [DW]), the former based on Hironaka's Desingularization, and the latter on Puiseux's Theorem. But, in general, these tools cannot be applied to sets belonging to o-minimal structures (e.g., to the set $\{(x, y) \in \mathbf{R}^2\colon y = \exp(-1/x), x > 0\}$ in the structure generated by the exponential function). Section 2 is devoted to the study of stratifications of definable functions. In general, definable functions cannot be stratified to satisfy the strict Thom condition (w_f). However, if $\mathcal{D}$ is polynomially

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