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GOLOMB'S SELF-DESCRIBED SEQUENCE AND FUNCTIONAL DIFFERENTIAL EQUATIONS

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A sequence (word) W of positive integers is *self-described* or *self-generating* if $\tau(W) = W$, where $\tau(W)$ is the sequence consisting of the numbers of consecutive equal entries of W. A famous self-generating bounded sequence is Kolakoski's $\underbrace{1,}_{1,}\ \underbrace{2,2,}_{2,}\ \underbrace{1,1,}_{2,}\ \underbrace{2,}_{1,}\ \underbrace{1,}_{1,}\ \underbrace{2,2,}_{2,}\cdots$ (see [Ch]). In this paper we consider Golomb's sequence F, which is the only nondecreasing self-generating sequence taking all positive integral values, $\underbrace{1,}_{1,}\ \underbrace{2,2,}_{2,}\ \underbrace{3,3,}_{2,}\ \underbrace{4,4,4}_{3,}\ \underbrace{5,5,5,}_{3,}\ \underbrace{6,6,6,6,}_{4,}\cdots$. Let ϕ denote the golden number. We prove that

$$F(n) = \phi^{2-\phi} n^{\phi-1} + \frac{n^{\phi-1}}{\log n} h\left(\frac{\log\log n}{\log\phi}\right) + O\left(\frac{n^{\phi-1}}{\log^2 n}\log\log n\right),$$

where the real function h is continuous and satisfies $h(x) = -h(x+1)$ $(x \geq 0)$. The method of proof is intimately connected with the more general problem of characterising the solution E_1 of an approximate functional integral equation of the type

$$E_1(t) = -\phi^{1-\phi} t^{\phi-2} \int_2^{\phi^{2-\phi} t^{\phi-1}} E_1(u)\, du + O\left(\frac{t^{\phi-1}}{\log^2 t}\right),$$

which we discuss in the second part of the paper.

1. Introduction

In the problem section of the American Mathematical Monthly in 1966, S.W. Golomb [Go] considered the unique nondecreasing sequence $\{F(n)\}_{n\geq 1} = \{1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 6, 7 \ldots\}$ "self-described" by the two conditions $F(1) = 1$ and $F(n) = |\{m : F(m) = n\}|$ $(n \geq 1)$. At the time he only requested an asymptotic expression for $F(n)$ as $n \to \infty$. We have

$$F(n) = cn^{\phi-1} + E(n), \tag{1.1}$$

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