ILLINOIS JOURNAL OF MATHEMATICS
Volume 30, Number 2, Summer 1986

DEFINABILITY IN THE TURING DEGREES[1]

BY

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In Memoriam W.W. Boone

1. Introduction

Definability has provided the most fruitful approach to understanding the model-theoretic structure of $\mathscr{D}$, the Turing degrees ordered by Turing reducibility.

The origin of this approach is Spector's result [9] that every countable ideal in $\mathscr{D}$ is uniformly definable from parameters in $\mathscr{D}$. Spector's theorem also shows that the first order theory of $\mathscr{D}$ includes an interpretation of quantification over such ideals.

Simpson [8] used this result, the embedding theorems for upper semi-lattices as initial segments of $\mathscr{D}$, and a coding of models of arithmetic as initial segments of $\mathscr{D}$, to show that there is a faithful interpretation of second order arithmetic in the first order theory of $\mathscr{D}$. In this interpretation, second order quantification over the coded model of arithmetic is interpreted by quantification over ideals in $\mathscr{D}$.

Nerode and Shore [4], [5] gave a simplified way to code models by initial segments. With their method of coding, the degree of the code of a set of integers is close to the degree of the set itself. They applied their method of coding to show that every automorphism of $\mathscr{D}$ with a predicate for the arithmetic degrees is the identity on a cone of degrees. Subsequently, see [2, Harrington-Shore] and [3, Jockusch-Shore], the arithmetic degrees were shown to be definable in $\mathscr{D}$. The best result currently known is that every automorphism is the identity on the cone above 0^{ω} and maps every degree to one arithmetic in it. In further work, Shore [7] showed that the relation "$\vec{c}$ is a code for an element of x" is first order definable in $\mathscr{D}$ for those x above 0^{ω}. This last result explains why every automorphism is the identity above 0^{ω}. Namely, both $\vec{c}$ and its isomorphic image must code the same set of integers, so the degree of the set coded by $\vec{c}$ must be fixed by any isomorphism. Shore's result

Received May 24, 1985.

[1] The authors were partially supported by grants from the National Science Foundation. Thanks are due to Klaus Ambos-Spies who pointed out the application of Welch's Theorem in Section 4, and to Marcia Groszek for reviewing the preliminary version of this paper.