ILLINOIS JOURNAL OF MATHEMATICS
Volume 30, Number 2, Summer 1986

DEFINABLE SUBGROUPS OF THE PRODUCT OF TWO GROUPS

BY

ALI NESIN

In Memoriam W.W. Boone

In this article we investigate definable subgroups of the product of two groups. Our language contains only function symbols together with symbols of the theory of groups. Such a language can be realized in the product of two structures in an obvious way. For the sake of simplicity a structure for such a language will be called a group. In this article definable will always mean definable with parameters. If G and H are two groups pr_1 (resp. pr_2) will denote the obvious projection map $G \times H \to G$ (resp. $G \times H \to H$). We prove the following result:

THEOREM. *Let G and H be two groups. Let K be a definable subgroup of $G \times H$. Then there are $A \lhd pr(K)$, $B \lhd pr(K)$ and $g_1, \ldots, g_n$ in G, $h_1, \ldots, h_n$ in H such that*

(1) *A and B are definable subgroups of G and H respectively,*
(2) *$A \times B \subseteq K$ and $[K: A \times B]$ is finite,*
(3) *$K = (A \times B) \cup (g_1 A \times h_1 B) \cup \cdots \cup (g_n A \times h_n B)$,*
(4) *$g_i A \cap g_j A = h_i B \cap h_j B = \varnothing$ if $i \neq j$.*

Furthermore such A and B are unique.

If a group G has a minimal definable subgroup of finite index then it is unique. We call it the connected component of G and denote it G^0. If G is ω-stable or ω_0-categorical and stable then the connected component always exists (see [1]).

COROLLARY. *Suppose G^0, H^0 exist. Then $(G \times H)^0$ exists and $G^0 \times H^0 \subseteq (G \times H)^0$.*

Proof. Let C be a definable subgroup of $G \times H$ of finite index. By the theorem there are definable subgroups A and B of G and H respectively such that $A \times B \subseteq C$ and $[C: A \times B]$ is finite. Therefore $[G \times H: A \times B]$, and hence also $[G: A]$, $[H: B]$ are finite. This shows that $G^0 \subseteq A$, $H^0 \subseteq B$, so

Received April 30, 1985.

© 1986 by the Board of Trustees of the University of Illinois
Manufactured in the United States of America