ILLINOIS JOURNAL OF MATHEMATICS
Volume 32, Number 1, Spring 1988

INVARIANT SUBSPACE LATTICES THAT COMPLEMENT EVERY SUBSPACE

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1. Introduction and the main result

In the theory of factorizations of operator polynomials and analytic operator functions (see [4] [5] [8] [9]) it is necessary to study invariant subspaces of a given operator (acting on a complex Hilbert space) that are direct complements to a given subspace. Of course, such invariant subspaces do not always exist. (Easy finite dimensional examples bear this out.) The following fact (due to D. Gurarie [6]) proved to be a very useful tool in the factorization theory (see [9] [10]): Let A be a (bounded linear) operator acting on a Hilbert space $\mathscr{H}$, which is similar to a normal operator with finite spectrum. Then for every (closed) subspace $\mathscr{M}$ of $\mathscr{H}$ there is an A-invariant subspace $\mathscr{R}$ such that $\mathscr{M} \cap \mathscr{R} = \{0\}$ and $\mathscr{M} + \mathscr{R} = \mathscr{H}$.

The main goal of this article is to prove that the operators similar to normals with finite spectrum are the only ones whose lattice of invariant subspaces complements every subspace.

It is convenient to introduce some definitions and notation. A *subspace* of a (complex) Hilbert space $\mathscr{H}$ is a closed linear manifold. Two subspaces, $\mathscr{M}$ and $\mathscr{R}$, are called *complementary* (denoted $\mathscr{M} \dot{+} \mathscr{R} = \mathscr{H}$) if $\mathscr{M} \cap \mathscr{R} = \{0\}$, $\mathscr{M} + \mathscr{R} = \mathscr{H}$. An *operator* $A: \mathscr{H} \to \mathscr{H}$ is a bounded linear transformation from $\mathscr{H}$ into itself, and we denote by $\mathscr{L}(\mathscr{H})$ the algebra of all operators acting on $\mathscr{H}$. The lattice of all invariant subspaces of $T \in \mathscr{L}(\mathscr{H})$ is denoted by Lat T. We shall say that Lat T has the *complement property* if for every subspace $\mathscr{M}$ and $\mathscr{H}$ there exists $\mathscr{R} \in \operatorname{Lat} T$ such that $\mathscr{M} \dot{+} \mathscr{R} = \mathscr{H}$. The lattice Lat T is said to have the *chain complement property* if for every finite chain of subspaces

$$\mathscr{M}_1 \subset \mathscr{M}_2 \subset \cdots \subset \mathscr{M}_r \subset \mathscr{H}$$

there exist a chain

$$\mathscr{R}_1 \supset \mathscr{R}_2 \supset \cdots \supset \mathscr{R}_r$$

of T-invariant subspaces such that $\mathscr{M}_i \dot{+} \mathscr{R}_i = \mathscr{H}$ for all $i = 1, 2, \ldots, r$.

Received September 3, 1986.
[1] Research partially supported by grants from the National Science Foundation.