ILLINOIS JOURNAL OF MATHEMATICS
Volume 32, Number 1, Spring 1988

GENERATORS AND RELATIONS FOR FINITELY GENERATED GRADED NORMAL RINGS OF DIMENSION TWO

BY

FRANCES VAN DYKE

Chapter 1. Introduction

Assume that R is a finitely generated graded normal ring of dimension 2 over **C** such that $R = \oplus_k R_k$ where $R_k = 0$ if $k < 0$ and $R_0 = \mathbf{C}$. This implies that R is the coordinate ring of a normal affine surface which admits a $\mathbf{C}^*$-action with a unique fixed point P, corresponding to the maximal ideal $\oplus_{k>0}^{\infty} R_k$ (see [5]). Henry Pinkham has shown that R is isomorphic to $\mathscr{L}(D) = \oplus_{n=0}^{\infty} L(nD)$ where D is a divisor on a Riemann surface X of genus g of the form

$$D = \sum_{p \in X} n_p P + \sum_{\substack{i=1 \\ p_i \in X}}^{k} \left(\frac{\beta_i}{\alpha_i}\right) P_i \qquad (*)$$

where $n_p \in Z$, all but finitely many $n_p = 0$, $0 < \beta_i/\alpha_i < 1$, and $L(nD)$ denotes the set of meromorphic functions f, such that $\operatorname{div}(f) + nD \geq 0$. It is easily seen that for each n, $L(nD)$ is a vector space over C.

It is always possible to choose a minimal set $S = \{y_1, \ldots, y_k\}$ of generators for $\mathscr{L}(D)$ such that the elements of S are homogeneous i.e. $y_j \in L(q_j D)$ for some q_j. In the polynomial ring $C[Y_1, \ldots, Y_k]$ give the variable Y_i degree q_i; then there exists a graded surjective homomorphism

$$\varphi\colon C[Y_1, \ldots, Y_k] \to \mathscr{L}(D), \quad \varphi(Y_i) = y_i.$$

Let I be the kernel of φ. We call I the ideal of relations for $\mathscr{L}(D)$ corresponding to S.

In the following paper it is shown that in many cases a minimal set of homogeneous generators S and generators for the corresponding ideal of relations I for $\mathscr{L}(D)$ can be determined if homogeneous generators and relations are known for $\mathscr{L}(D_1)$ where $D_1 < D$ and $\mathscr{L}(D_1)$ has a much simpler

Received July 25, 1986.