ILLINOIS JOURNAL OF MATHEMATICS
Volume 32, Number 4, Winter 1988

DUALITY IN SPACES OF OPERATORS AND SMOOTH NORMS ON BANACH SPACES

BY

GILLES GODEFROY AND PIERRE DAVID SAPHAR

Introduction

This work deals with the spaces of operators between Banach spaces and their duality, from an infinite-dimensional point of view. We use isomorphic as well as isometric tools. In particular we investigate and use the fruitful interplay between metric and weak topological properties of Banach spaces. Let us summarize the content of this paper.

In Section 1 we use the technique of [6] for obtaining a general representation (1.3) of the space $K(X, Y)^{**}$, X and Y being reflexive spaces. Our method leads to an improvement (1.5) of a classical result of A. Grothendieck, and of a result (1.6) of [3].

In Section 2 we define and use the unique extension property (U.E.P.) which turns out to be the natural tool for lifting the M.A.P. from E to E^* (2.2). A geometrical lemma (2.4) enables us to find a usable condition for obtaining the U.E.P. (2.5).

In Section 3 we show that many spaces have the U.E.P. However, the class is not stable by 1-complemented subspaces (3.1). We find a surprising characterization (3.3) of the dual norms on the James space. We notice that there is a space with a Frechet-differentiable norm but no equivalent Hahn-Banach smooth norm (3.4), and we present a renorming problem.

Section 4 presents an isomorphic version (4.3) of the results of Section 2. We obtain, in particular, an extension of a result of [35].

In Section 5 we use the smoothness of the norm of $K(X, Y)$ (X and Y reflexive) for showing that such a space is "far" from being a dual space if it is not reflexive ((5.2) and its corollaries)). This improves Theorem 2 of [6]. We show also that if X is reflexive, the space $L(X)$, equipped with the operator norm, has an unique isometric predual (5.11).

This work contains many examples which show as far as possible that our results are sharp. Let us mention that, unless otherwise specified, the Banach

Received November 24, 1986.