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THE 4-CLASS RANKS OF QUADRATIC EXTENSIONS OF CERTAIN IMAGINARY QUADRATIC FIELDS

BY

FRANK GERTH III

1. Introduction and statement of main result

Let K be a quadratic extension of the field of rational numbers $\mathbf{Q}$. Let C_K be the 2-class group of K in the narrow sense. Then it is a classical result that rank $C_K = t - 1$, where t is the number of primes that ramify in $K/\mathbf{Q}$. Let R_K be the 4-class rank of K in the narrow sense; i.e.,

$$R_K = \operatorname{rank} C_K^2 = \dim_{\mathbf{F}_2}\left(C_K^2/C_K^4\right).$$

Here $\mathbf{F}_2$ is the finite field with two elements, and C_K^2/C_K^4 is an elementary abelian 2-group which we are viewing as a vector space over $\mathbf{F}_2$. In [6] we have presented results which specify how likely it is for $R_K = 0, 1, 2, \ldots,$ both for imaginary quadratic extensions of $\mathbf{Q}$ and for real quadratic extensions of $\mathbf{Q}$.

Suppose now we replace the base field $\mathbf{Q}$ by an imaginary quadratic field F whose class number is odd, and suppose K is a quadratic extension of F. We let C_K denote the 2-class group of K. Then rank $C_K = t - 1 - \beta$, where t is the number of primes that ramify in K/F, and $\beta = 0$ or 1. (See Equation 3.5 for more details.) We let R_K denote the 4-class rank of K, and we ask the following question: how likely is $R_K = 0, 1, 2, \ldots$? Since the 2-class groups of both F and $\mathbf{Q}$ are trivial, and since the groups of units in the rings of integers of F and $\mathbf{Q}$ are finite cyclic groups, there is a reasonable expectation that the 4-class ranks of quadratic extensions of F will exhibit a behavior similar to the 4-class ranks of quadratic extensions of $\mathbf{Q}$.

To make the situation more precise, we introduce some notation. We let $\mathcal{O}_F$ denote the ring of integers of F. For a nonzero ideal I of $\mathcal{O}_F$, we let $N(I)$ denote the absolute norm of I. Equivalently $N(I) = [\mathcal{O}_F\colon I]$. For a quadratic extension K of F, we let $D_{K/F}$ denote the relative discriminant. For each

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