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HAUSDORFF DIMENSION AND PERRON-FROBENIUS THEORY

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1. Introduction

In this paper we describe a method of calculating Hausdorff dimension of certain subsets of the unit interval, using the Perron-Frobenius theory of non-negative matrices. The sets in question are as follows. Let $b > 1$ be a fixed integer. Each $x \in (0, 1)$ can then be expressed in base b as

$$x = \sum_{n=1}^{\infty} e_n(x) b^{-n} = 0.e_1(x)e_2(x)\dots, \tag{1}$$

where $0 \le e_n(x) \le n - 1$. The functions $e_n(x)$ are called the digits of x in base b. If we stipulate that the e_n's have the property that for each x, $e_n(x) < b - 1$ for infinitely many n's, then the expansion in (1), i.e., all the functions $e_n(x)$, is uniquely determined. The lack of uniqueness is an issue only for countably many x's. Now, given two integers $0 \le c \le r$ we define the set $T_b(c, r)$ to be

$$T_b(c, r) = \left\{ x \in (0,1)\colon \sum_{j=1}^{r} e_{n+j}(x) \ge c,\ n = 0, 1, 2\dots \right\}.$$

In other words, $T_b(c, r)$ consists of those x's in $(0, 1)$, for which any r consecutive base b digits sum up to at least c. We will show how to calculate the Hausdorff dimension of these sets. The interest in them arose from the paper [2] by one of the authors, in which a Fibonacci type of recurrence of sets was studied. The set arising in that paper was $T_2(1, 2)$, the Hausdorff dimension of which turns out to be $\log_2(\frac{1}{2}(1 + \sqrt{5}))$. In order to keep the exposition and notation clear we restrict our attention to case $b = 2$, i.e., to the binary expansion. Extension of the method to arbitrary b's is completely routine.

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