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QUASI-HEREDITARY ALGEBRAS

BY

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Dedicated to the memory of Irving Reiner

In their work on highest weight categories arising in the representation theory of Lie algebras and algebraic groups, E. Cline, B. Parshall and L. Scott recently introduced the notion of a quasi-hereditary algebra (see [1] and [2]). They define a quasi-hereditary algebra recursively in terms of the existence of a particular idempotent ideal; finite-dimensional hereditary algebras are typical examples of quasi-hereditary algebras. On the other hand, they showed that every quasi-hereditary algebra has finite global dimension.

The purpose of this note is to establish the following three results. First, finite-dimensional hereditary algebras are characterized as those quasi-hereditary algebras which satisfy a certain refinement property on chains of their idempotent ideals (Theorem 1). Second, all finite-dimensional algebras of global dimension 2 are shown to be quasi-hereditary (Theorem 2). Third, the question of whether every finite-dimensional algebra of finite global dimension is quasi-hereditary is answered in the negative by providing an example of an (11-dimensional serial) algebra of global dimension 4 which is not quasi-hereditary. The same example illustrates that the class of quasi-hereditary algebras is not closed under tilting (in the sense of [4]).

In what follows, all rings are semiprimary rings. An associative ring A with 1 is called *semiprimary* if its Jacobson radical N is nilpotent and A/N is semisimple artinian. Recall that an ideal I of A is idempotent if and only if $I = AeA$ for an idempotent e of A; in particular, I is a minimal (non-zero) idempotent ideal provided that e is primitive. An ideal J of A is said to be a *heredity* ideal of A if $J^2 = J$, $JNJ = 0$ and J, considered as a right A-module J_A, is projective. In fact, this also implies that the left A-module ${}_AJ$ is projective (see [2] or [3]). A semiprimary ring A is called *quasi-hereditary* if there is a chain

$$0 = J_0 \subset J_1 \subset \cdots \subset J_{t-1} \subset J_t \subset \cdots \subset J_m = A$$

of ideals of A such that, for any $1 \leq t \leq m$, J_t/J_{t-1} is a heredity ideal of A/J_{t-1}. Such a chain of idempotent ideals is called a *heredity chain*. Let us

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