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WEIL'S GROUP CHUNK THEOREM: A TOPOLOGICAL SETTING

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Introduction

A. Weil showed that a birational group law that is only partially defined can be extended to an algebraic group [15], [16]. We show below (§1) that a similar construction can be carried out in a topological setting, where the topology is not necessarily the Zariski topology. In §2 we enrich our topological spaces with sheaves and prove a version of Weil's theorem for this "structured" setting. We then derive Weil's original theorem as well as two variations, covering the cases of "quasi-algebraic" group chunks and "differentially algebraic" group chunks (§3). We note that our theorem, though quite general, does not include the scheme theoretic version of Weil's theorem given in [1].

Our version of Weil's theorem can be applied to resolve a problem arising in model theory. The question whether all groups that are first order definable in algebraically closed fields are isomorphic to algebraic groups is connected with work of Cherlin, Poizat, and Zil'ber (cf. [10, [11]). As we show here (§4) a positive solution follows from the group chunk theorem. In characteristic 0, Weil's theorem suffices, and in characteristic $p > 0$ the quasi-algebraic version is needed, together with a theorem of Serre [13] characterizing quasi-algebraic groups.

Another approach to the latter problem was given by Hrushovski (unpublished); there are expositions of his treatment in [2], [12]. It is similar in spirit to our approach, with two variations: since in this application the abstract group is already given to us as a definable group, he omits the first step in the proof of Weil's theorem, and passes directly to the introduction of a topology and structure sheaf; secondly where we combine our generalized Weil theorem with a result of Serre, Hrushovski uses the idea of Serre's proof to reduce to an algebraic group chunk. Hrushovski also gave a generalization of Weil's group chunk theorem in quite a different direction in his thesis [4].

Our group chunk theorem is inspired by unpublished notes of W. van der Kallen [5], and in particular we follow his approach to the construction of the enveloping group as an abstract group in §1. In topologizing the group and equipping it with a sheaf we adopt a different approach which works in greater generality.

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