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HOLOMORPHIC FUNCTIONS WITH POSITIVE REAL PART ON THE UNIT BALL OF C^n

BY

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Consider the set $\mathscr{P}$ of holomorphic functions on the open unit ball B of C^n which have positive real part and take the value 1 at 0. Except in the case where $n = 1$, the problem of identifying the extreme elements of the convex set $\mathscr{P}$ is unsolved. Some results on this interesting and natural question have been obtained by Forelli in papers mentioned below and there is a discussion of it in the book of Rudin [7]. It seems, however, that a complete and satisfactory solution is not close at hand.

In this paper we study the relationship between the extreme elements of $\mathscr{P}$ and the extreme elements of the closed unit ball $\mathscr{U}$ of the space $H^\infty(B)$ via the representation

$$f(z) = (1 + g(z))/(1 - g(z)), \tag{1}$$

where g is a member of $\mathscr{U}$ which vanishes at 0. Forelli has shown that the function (1) is an extreme point of $\mathscr{P}$ in the cases where

$$g(z) = g(z_1, z_2, \ldots, z_n) = z_1^2 + z_2^2 + \cdots + z_n^2$$

and

$$g(z) = cz^\alpha = cz_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n},$$

where the greatest common divisor of the positive integers α_j is 1 and c is a constant chosen so that

$$\|g\| = \sup\{|g(z)| : z \in B\} = 1.$$

See [1], [3]. Forelli has also produced sufficient conditions on a homogeneous polynomial p in order that $(1 + p)/(1 - p)$ be extreme in $\mathscr{P}$ [3]. One of our main results implies that, if g is a homogeneous polynomial of degree $k \geq 1$ which is also an extreme point of $\mathscr{U}$, then there exists a polynomial r of degree $\leq k - 1$ such that $(1 + g + r)/(1 - g)$ is an extreme point of $\mathscr{P}$. We also use our results to derive the examples of Forelli described above, as well as some new examples of extreme members of $\mathscr{P}$.

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