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INTEGRABLE CONNECTIONS RELATED TO MANIN AND SCHECHTMAN'S HIGHER BRAID GROUPS

BY

TOSHITAKE KOHNO

To the memory of K. T. Chen

Introduction

In [11], Manin and Schechtman introduced a series of hyperplane arrangements called the discriminantal arrangements, which generalize the braid arrangements. For positive integers $k \leq n$, the discriminantal arrangement $\mathscr{B}(n, k)$ consists of hyperplanes D_J in $\mathbf{C}^n$ labelled with $J \subset \{1, \dots, n\}$ such that $|J| = k$. The complementary space $U(n, k) = \mathbf{C}^n - \bigcup_J D_J$ parametrizes affine hyperplanes $(H_1, \dots, H_n)$ in $\mathbf{C}^k$ which are in general position and each H_i is parallel to H_i^0 for some fixed affine hyperplanes $(H_1^0, \dots, H_n^0)$ in general position. The fundamental group of $U(n, k)$ is called a higher braid group and coincides with the pure braid group with n strings if $k = 1$. The combinatorial aspects of such arrangements were studied by Manin and Schechtman [11].

Combining with the theory of K. T. Chen [2], [3], we see that the completion of the group ring of the higher braid group over $\mathbf{C}$ with respect to the powers of the augmentation ideal is generated by X_J, which are in one-to-one correspondence with the hyperplanes D_J, with relations

(i) $[X_J, \Sigma_{I \subset K} X_I] = 0$ for any $K \subset \{1, \dots, n\}$ with $|K| = k + 2$,

(ii) $[X_{J_1}, X_{J_2}] = 0$ if $|J_1 \cup J_2| \geq k + 3$.

These relations generalize the infinitesimal pure braid relations in the sense of [9] and [10]. They give the integrability condition of the connection of the form $\Sigma_J X_J \, d \log \varphi_J$ where X_J is a constant matrix and φ_J is a linear form with $\operatorname{Ker} \varphi_J = D_J$. In the case $k = 1$ the above infinitesimal pure braid relations were studied in relation with the classical Yang-Baxter equation and we obtain one parameter family of linear representations of the pure braid groups as the holonomy of this connection for any simple Lie algebra and its representations (see [10]). The purpose of this note is to give a generalization of this construction to the higher braid groups. Namely, given a finite dimensional

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