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TRANSCENDENTAL ASPECTS OF THE RIEMANN-HILBERT CORRESPONDENCE[1]

BY

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In memory of Kuo-Tsai Chen

1. Introduction

Given a system of ordinary differential equations, locally with coefficients which are holomorphic functions on a Riemann surface, one obtains a representation of the fundamental group. If P is a chosen base point, and if γ is a path beginning and ending at P, then there is a matrix $m(\gamma)$ which expresses the transformation effected on a basis of solutions at P, by the process of continuation around γ. Thus there is a map from the set of systems of differential equations to the set of representations—a map which has come to be known as the Riemann-Hilbert correspondence. The purpose of this paper is to describe some properties of this map which reflect on its essentially transcendental nature. The main technique is Kuo-Tsai Chen's expansion of the solution of a system of differential equations as a sum of iterated integrals [3], [4]. I never met K-T. Chen, but learned about his work from Richard Hain. I hope that this paper may make a contribution toward showing the influence of Chen's ideas.

In order to illustrate the types of problems to be considered, let us discuss the case of systems of rank one on a compact Riemann surface X of genus g. A system of rank one consists of a line bundle L and a connection ∇ on L. The set of these objects forms a group under tensor product, and we will denote this group by U. It is an algebraic group. There is a map to the Jacobian of line bundles on X of degree zero, and the kernel is the set of connections on the trivial bundle:

$$0 \to H^0(\Omega^1_X) \to U \to \mathrm{Jac}(X) \to 0.$$

On the other hand, the set of one dimensional representations of the fundamental group of X is $\mathrm{Hom}(\pi_1, \mathbf{G}_m)$, which is isomorphic to $\mathbf{G}_m^{2g}$ after a choice of generators $\gamma_1, \ldots, \gamma_{2g}$. The Riemann-Hilbert correspondence in this case is

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