ILLINOIS JOURNAL OF MATHEMATICS
Volume 34, Number 2, Summer 1990

THE HOMOTOPY INVARIANCE OF THE KURANISHI SPACE

BY

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In memory of Kuo-Tsai Chen

This paper is our second paper devoted to applying the ideas of rational homotopy theory as developed by Chen, Quillen and Sullivan to deformation problems from analytic geometry. Our first paper [GM1] studied the deformations of flat connections and holomorphic structures on principal bundles (for the most part over compact Kähler manifolds). In this paper we study the deformation spaces of complex structures on compact manifolds. The "controlling differential graded Lie algebra" (L, d) of Deligne, [GM1, p. 48], is now the Kodaira-Spencer algebra

$$(L, d) = \left(\bigoplus_{q=0}^{\infty} \mathscr{A}^{0,q}(M, T^{1,0}), \bar{\partial} \right)$$

where $\mathscr{A}^{0,q}(M, T^{1,0})$ denotes the space of C^{∞} exterior differential forms on M of type $(0, q)$ with values in the holomorphic tangent bundle.

In [Ku1] and [Ku2], Kuranishi constructed the versal deformation of a compact complex manifold M (see the appendix of this paper for definitions and terminology). The parameter (base) space of this deformation is an analytic germ in $H^1(L)$ with base point $0 \in H^1(L)$ which we will denote $(\mathscr{K}, 0)$ or $\mathscr{K}$ and will call the Kuranishi space. Although over twenty five years have passed since [Ku1] appeared many basic questions concerning $\mathscr{K}$ remain unanswered. In this paper we prove that $\mathscr{K}$ is a "homotopy invariant" of L and use this principle to compute $\mathscr{K}$ in the examples detailed below. We now explain precisely what we mean by the "homotopy invariance" of $\mathscr{K}$.

Let (L, d) be a differential graded Lie algebra over a field $\mathbf{k}$ (either $\mathbf{C}$ or $\mathbf{R}$ in what follows). Choose a complement $C^1(L)$ to the 1-coboundaries $B^1(L) \subset L^1$. We define a functor $A \to Y_L(A)$ on the category of Artin local $\mathbf{k}$-algebras by

$$Y_L(A) = \{\eta \in C^1(L) \otimes \mathfrak{m} \colon d\eta + \tfrac{1}{2}[\eta, \eta] = 0\}$$

Here $\mathfrak{m}$ is the maximal ideal of the Artin local $\mathbf{k}$-algebra A. It is proved in §1 that the functor Y_L satisfies the hypotheses of Theorem 2.11 of [Sc] and is

Received February 28, 1989.