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A_∞-ALGEBRAS AND THE CYCLIC BAR COMPLEX

BY

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In memory of Kuo-Tsai Chen

This paper arose from our use of Chen's theory of iterated integrals as a tool in the study of the complex of S^1-equivariant differential forms on the free loop space LX of a manifold X (see [2]). In trying to understand the behaviour of the iterated integral map with respect to products, we were led to a natural product on the space of S^1-equivariant differential forms $\Omega(Y)[u]$ of a manifold Y with circle action, where u is a variable of degree 2. This product is not associative but is homotopy associative in a precise way; indeed there is whole infinite family of "higher homotopies". It turns out that this product structure is an example of Stasheff's A_∞-algebras, which are a generalization of differential graded algebras (DGAs).

Using the iterated integral map, it is a straightforward matter to translate this product structure on the space of S^1-equivariant differential forms on LX into formulas on the cyclic bar complex of $\Omega(X)$. Our main goal in this paper is to show that in general, the cyclic bar complex of a commutative DGA A has a natural A_∞-structure and we give explicit formulas for this structure. In particular, this shows that the cyclic homology of A has a natural associative product, but it is a much stronger result, since it holds at the chain level. Thus, it considerably strengthens the results of Hood and Jones [3].

We also show how to construct the cyclic bar complex of an A_∞-algebra, and in particular define its cyclic homology. As hinted at in [2], this construction may have applications to the problem of giving models for the $S^1 \times S^1$-equivariant cohomology of double loop spaces $LL(X)$ of a manifold and, since the space of equivariant differential forms on a smooth S^1-manifold Y is an A_∞-algebra, to the problem of finding models for the space of $S^1 \times S^1$-equivariant differential forms on LY. Although the methods that we use were developed independently, they bear a strong resemblance with those of Quillen [6].

Finally, we discuss in our general context the Chen normalization of the cyclic bar complex of an A_∞-algebra. This is a quotient of the cyclic bar complex by a complex of degenerate chains which is acyclic if A is connected,

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[1]In the preprint of [2], the maps m and $\tilde{m}$ are exchanged, for which we beg the reader's forgiveness.