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THE EMBEDDING OF BANACH SPACES INTO SPACES WITH STRUCTURE

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1. Introduction

Let X be a separable Banach space. A sequence $\{Y_n\}_{n=1}^{\infty}$ of finite dimensional subspaces of X is called a *finite dimensional decomposition* (f.d.d., in short) of X if each $x \in X$ has a unique representation $x = \sum_{n=1}^{\infty} T_n x$ with $T_n x \in Y_n$. A basis of X is a f.d.d. where each Y_n is of dimension 1. It is well known and easy to prove that X has a f.d.d. if and only if there is a sequence $\{P_n\}_{n=1}^{\infty}$ of commuting projections on X such that each P_n is of finite rank, $\sup_n \|P_n\| < \infty$, $P_1 X \subset P_2 X \subset \cdots$ and $\bigcup_n P_n X$ is dense in X. The existence of Banach spaces without the approximation property makes it reasonable to investigate how "close" a given separable space is to spaces with a f.d.d. In this direction are the following three problems (the first two of which were previously solved (see [2] and [5])).

PROBLEM 1. *Given a separable Banach space X does there exist a subspace E of X such that both E and X/E have f.d.d.'s?*

PROBLEM 2. *Given a separable Banach space E does there exist a separable space X and a subspace Y of X, both with an f.d.d., such that $E = X/Y$?*

PROBLEM 3. *Given a separable space E does there exist a space X containing E such that both X and X/E have f.d.d.'s?*

The first problem is positively solved by W. B. Johnson and H. P. Rosenthal in [2]. The second one is answered by J. Lindenstrauss in [5] in the following strong sense: every separable space E is isomorphic to a quotient X^{**}/X where both X and X^{**} have bases. The purpose of this paper is to give a positive solution of Problem 3. Since every complemented subspace of a space with an f.d.d. has the bounded approximation property one does not expect a given separable space to be complemented in a space with an f.d.d.

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