ILLINOIS JOURNAL OF MATHEMATICS
Volume 34, No. 3, Fall 1990

SOME OPERATOR INEQUALITIES CONCERNING GENERALIZED INVERSES

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In this paper we extend to the von Neumann-Schatten classes some inequalities concerning generalized inverses originally established by Penrose [7] for matrices.

It is well known that if T is any matrix then there exists a matrix T^- such that $TT^-T = T$. The matrix T^- is called a generalized (or, sometimes, a pseudo or inner) inverse of T. A non-invertible matrix has an infinity of generalized inverses, but an invertible matrix has a unique generalized inverse which coincides with its ordinary inverse.

Although the matrix T may have an infinity of generalized inverses, there exists a canonical generalized inverse, called the Moore-Penrose inverse and denoted by T^+, which is uniquely determined by T. Further, as Penrose showed in [7], the Moore-Penrose inverse satisfies the following inequalities: for all X,

$$\|AX - C\|_2 \geq \|AA^+C - C\|_2 \tag{P.1}$$

with equality occurring in (P.1) if and only if $X = A^+C + (I - A^+A)L$ where L is arbitrary; and

$$\|A^+C + (I - A^+A)L\|_2 \geq \|A^+C\|_2 \tag{P.2}$$

with equality occurring in (P.2) if and only if $(I - A^+A)L = 0$. (The only restrictions on the matrices occurring in (P.1) and (P.2) is that they be conformable for multiplication. In (P.1) and (P.2) $\|\cdot\|_2$ denotes the Euclidean norm on matrices.)

This paper contains, in Theorems 2.1 and 2.2, infinite-dimensional extensions of the inequalities (P.1) and (P.2) to the supremum norm on $\mathscr{L}(H)$ and to the von Neumann-Schatten norms $\|\cdot\|_p$, where $2 \leq p < \infty$. The proofs of Theorems 2.1 and 2.2 are an extension of Penrose's original proof of (P.1) and (P.2) and depend on an inequality, viz. Theorem 1.7, about operators having orthogonal ranges/co-ranges.

Received February 29, 1988.