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FUNCTIONS WITH A UNIQUE MEAN VALUE

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Section 1

Let G be a Hausdorff locally compact group. An *admissible* subspace $S \subset L_\infty(G)$ is a subspace containing the constants such that if $f \in S$, then ${}_gf(x) = f(g^{-1}x)$ defines ${}_gf \in S$. A function $f \in L_\infty(G)$ *potentially has a unique left invariant mean* if there is a constant c such that whenever $f \in S \subset L_\infty(G)$, S an admissible subspace, then any left invariant mean M on S has $M(f) = c$. A function $f \in L_\infty(G)$ *has a unique left invariant mean value* if it potentially has a unique left invariant mean value, and also there is an admissible subspace $S \subset L_\infty(G)$ with $f \in S$ and there is a left invariant mean on S. If G is amenable, the above two notions are the same, but in general a function may potentially have a unique mean value without actually having one. The analogous notions for right translations or translations on left and right are easy to formulate.

A function $f \in L_\infty(G)$ *left averages* (*to* c) if there is a constant c in the $\|\cdot\|_\infty$- closed convex hull of $\{{}_gf: g \in G\}$. Any function which left averages to a constant must potentially have a unique left invariant mean value. The following is well known.

1.1. THEOREM. *If G is amenable as a discrete group, then the following are equivalent for $f \in L_\infty(G)$:*

(1) *f has a unique left invariant mean value*;

(2) *f left averages*;

(3) $f \in \|\cdot\|_\infty$*-closed span* $C \cup \{{}_gf - f: g \in G\}$;

(4) $f \in \|\cdot\|_\infty$*-closed span* $C \cup \{{}_g\zeta - \zeta: \zeta \in L_\infty(G), g \in G\}$.

Remark. The implications (2) implies (3) and (3) implies (4) are always true. The implications (3) implies (1), (2) implies (1) and (1) implies (4) only need the assumption that G is amenable as a locally compact group. However, all the other implications need the hypothesis that G is amenable as a discrete group. For example, if G is a compact group with a unique

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