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ON COANALYTIC FAMILIES OF SETS IN HARMONIC ANALYSIS

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I. Introduction

In the last few years many natural families of sets, or of functions, from harmonic analysis were shown to be Π_1^1-hard; the reader will find references in the recent book of A. Kechris and A. Louveau [7]. The goal of the present work is to invite another family to join the club.

Let us recall that a subset Λ of a discrete abelian group Γ is called a *Rosenthal set* [9, Def. 2.1] if $L_\Lambda^\infty(\hat{\Gamma}) = \mathscr{C}_\Lambda(\hat{\Gamma})$. It was shown by H. P. Rosenthal ([12]) that there are (what we call now) Rosenthal sets which are not Sidon. It follows from our main result that if Γ is a countably infinite abelian discrete group, then the family $\mathrm{Ros}(\Gamma)$ of Rosenthal subsets of Γ is a Π_1^1-hard subset of $\mathscr{P}(\Gamma)$. This result means in particular that there is no hope to obtain "positive" characterizations of Rosenthal sets, or that any characterization will be at least as complex as the definition.

Our proofs combine a result of F. Lust-Piquard which enables us to construct Rosenthal sets [10, Th. 3], together with a technique of V. Tardivel [13]; actually, our method provides a proof of Tardivel's result which is slightly simpler than the original one. Let us mention however that our proof uses a delicate result on spectral synthesis due to Loomis [8], which depends on a theorem of Bohr about almost-periodic functions.

Notation. Throughout this paper, Γ denotes an abelian discrete group and $G = \hat{\Gamma}$ its compact dual group. $\mathscr{P}(\Gamma)$ is the power set of Γ; if $\Lambda \in \mathscr{P}(\Gamma)$, $L_\Lambda^\infty(G)$ denotes the space of bounded measurable functions on G, with respect to the Haar measure dm of G, whose Fourier transforms vanish outside Λ. The spaces $\mathscr{M}_\Lambda(G)$, $L_\Lambda^1(G)$ and $C_\Lambda(G)$ are defined similarly. A subset Λ of Γ is called a Rosenthal set if $C_\Lambda(G) = L_\Lambda^\infty(G)$; $\mathrm{Ros}(\Gamma)$ denotes the family of Rosenthal sets. Λ is called a Riesz set if $\mathscr{M}_\Lambda(G) = L_\Lambda^1(G)$, and the family of Riesz sets is $\mathscr{R}(\Gamma)$. A subset A of a Polish space P is Σ_1^1 (i.e. analytic) if it is a continuous image of the Polish space $\mathbf{N}^{\mathbf{N}}$; a subset C of P

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