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FINITE 2-GROUPS OF CLASS 2 IN WHICH EVERY PRODUCT OF FOUR ELEMENTS CAN BE REORDERED[1]

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1. Introduction

If n is an integer greater than 1, then a group G belongs to the class P_n if every ordered product of n elements can be reordered in at least one way; in other words, to each n-tuple $(x_1, x_2, \ldots, x_n)$ of elements of G there corresponds a non-trivial element σ of the symmetric group Σ_n such that

$$x_1 x_2 \cdots x_n = x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}.$$

The union of the classes P_n, $n \geq 2$, is denoted by P. It was shown in [4] that P consists precisely of the finite-by-abelian-by-finite groups.

Clearly P_2 is the class of abelian groups, while $G \in P_3$ if and only if $|G'| \leq 2$ [3]. Graham Higman [6] characterised finite groups of odd order in P_4 and also proved that a group G with $G' \cong V_4$ (the 4-group) always belongs to P_4. Then in [8], improving a result in [1], it was shown that all P_4-groups are metabelian. Finally in [9] the non-nilpotent P_4-groups were classified and the nilpotent P_4-groups were shown to have class at most 4. We recall the details of these results in §2.

The present work is a further contribution to the classification of P_4-groups. We determine precisely which finite 2-groups of class 2 belong to P_4. Combining this work with the results of [9] it has been possible to classify *all* P_4-groups and a complete description by M. Maj and the present authors will appear elsewhere. The finite 2-groups of class 2, however, are most conveniently treated independently. If G is such a group in P_4, we shall see that G' has exponent at most 4. Our main results are:

THEOREM A. *Let G be a finite 2-group of class 2 with G' of exponent 4. Then $G \in P_4$ if and only if $G' \cong C_4$ and G has a subgroup B of index 2 with $|B'| = 2$.*

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