ILLINOIS JOURNAL OF MATHEMATICS
Volume 35, Number 3, Fall 1991

REARRANGEMENT TECHNIQUES IN MARTINGALE SETTING

BY

RUILIN LONG[1]

The concept of rearrangement function was introduced by Hardy-Littlewood [5] about sixty years ago. It played a remarkable role in Lorentz space theory and its related interpolation theory. But for a long time, people preferred the distribution function technique to the rearrangement one. It was Herz [6], Bennett-Sharpley [2] and Bagby-Kurtz [1], etc., who showed that there was no reason for this preference. In this article, we will study some examples to show what are the superiority or inferiority of the rearrangement technique in obtaining several typical inequalities in martingale theory.

Let $(\Omega, \mathscr{F}, \mu)$ be a complete probability space with $\{\mathscr{F}_n\}_{n\geq 0}$ a nondecreasing sequence of sub-σ-fields such that $\mathscr{F} = V_n \mathscr{F}_n$, and each $(\Omega, \mathscr{F}_n, \mu)$ is complete. $f = (f_n)_{n\geq 0}$ is said to be a martingale (with respect to $\{\mathscr{F}_n\}_{n\geq 0}$), if each $f_n \in L^1(\Omega, \mathscr{F}_n, \mu)$, and $E(f_{n+1}|\mathscr{F}_n) = f_n$, $\forall n$. The Doob maximal function and the square function of the martingale $f = (f_n)_{n\geq 0}$ are defined as

$$Mf = \sup_n |f_n|, \qquad M_n f = \sup_{k\leq n} |f_k|, \tag{1}$$

$$Sf = \left(\sum_0^\infty |\Delta_n f|^2 \right)^{1/2}, \qquad S_n f = \left(\sum_{k=0}^n |\Delta_k f|^2 \right)^{1/2}, \tag{2}$$

where $\Delta_k f = f_k - f_{k-1}$, $k \geq 1$, $\Delta_0 f = f_0$. In what follows, we make the convention that for any process $\lambda = (\lambda_n)_{n\geq 0}$, λ_{-1} is taken to be equal to 0, unless otherwise specified. Let f be a measurable function on $(\Omega, \mathscr{F}, \mu)$. Its distribution function, rearrangement function, and averaged rearrangement function are defined respectively as

$$\sigma_f(\lambda) = |\{\omega \in \Omega : |f(\omega)| > \lambda\}|_\mu = |\{|f| > \lambda\}|, \qquad \lambda > 0, \tag{3}$$

$$f^*(t) = \inf\{\lambda : \sigma_f(\lambda) \leq t\}, \qquad t > 0, \tag{4}$$

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds, \qquad t > 0. \tag{5}$$

Received December 19, 1989.
1980 Mathematics Subject Classification (1985 Revision). Primary 60G42; Secondary 60E15.
[1]Supported by the National Science Foundation of China.